Willmore spheres in Riemannian manifolds

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Introduction

NOTATION:

▶ \((M, g)\) 3-d Riemannian manifold (later also \(\dim(M) \geq 3\))

▶ \(\Sigma\) closed (compact, \(\partial \Sigma = \emptyset\)) 2-d surface

▶ \(f : \Sigma \rightarrow M\) immersion, \(\hat{g}\) induced metric on \(\Sigma\)

▶ \(A_{ij}\) = II fundamental form of \(f(\Sigma)\)

▶ \(H = \frac{1}{2} A_{ij} \hat{g}^{ij} = k_1 + k_2\) mean curvature

▶ \(A_{\circ ij}\) = \(A_{ij} - H \hat{g}^{ij}\) traceless II fundamental form

Question

Which are the best immersions \(f\) ?
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Question

Which are the best immersions $f$?
Classical special immersions

\[ H \equiv 0 \Rightarrow \text{MINIMAL immersion (\to critical point of Area)} \]

\[ A \equiv 0 \Rightarrow \text{TOTALLY GEODESIC immersion} \]

\[ A_0 \equiv 0 \Rightarrow k_1 = k_2 \text{TOTALLY UMBILIC immersion} \]

FACT: in general they may not exist.

Examples: minimal in \( \mathbb{R}^3 \) (by max. principle) or totally umbilical in Berger Spheres \([\text{Souam-Toubiana (Comm. Math. Helv. '09)}]\) and more general in a generic homogeneous spaces \([\text{Manzano-Souam (Preprint '13)}]\).
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W_{cnf}(f) := \int_{f(\Sigma)} |A^\circ|^2 = \text{Conf. Willmore funct.}
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Remark 1-

\((M, g)\) and \(\Sigma\) are fixed at the beginning, minimize in the immersion \(f\), if \((M, g) = (\mathbb{R}^3, \text{eucl})\) then by Gauss Bonnet Theorem

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W(f) = W_{cnf}(f) + 2\pi \chi E(\Sigma) = \frac{1}{2} E(f) + \pi \chi E(\Sigma)
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Conformal invariance

Theorem (Weiner '78)

$\text{W}_{\text{cnf}}$ is conformally invariant, i.e.

\[ \forall u \in C^\infty(M) \Rightarrow \text{W}_{\text{cnf}}(f)[u] = \text{W}_{\text{cnf}}(f) \]

where $\text{W}_{\text{cnf}}(f)[u]$ is the conformal Willmore functional evaluated on $f(\Sigma)$ immersed in $(M, g[u])$.

Remark $W$ is conformal invariant in $\mathbb{R}^3$ but not in a general manifold $\Rightarrow \text{W}_{\text{cnf}}$ is the "correct" Willmore functional from a conformal point of view.
**Theorem (Weiner ’78)**

$W_{cnf}$ is conformally invariant, i.e.

$$\forall u \in C^\infty(M) \text{ called } g[u] := e^{2u}g \Rightarrow W_{cnf}(f)[u] = W_{cnf}(f)$$

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Some literature about existence of minimizers or critical points

\[ W = \int H^2 \] in Euclidean Space, i.e. \((M, g) = (\mathbb{R}^3, \text{eucl})\):

- Strict global minimum on standard spheres \(S^\rho\) (Willmore ‘60):
  \[ \forall \Sigma, \forall f : \Sigma \to \mathbb{R}^3 \Rightarrow W(f) \geq 4\pi \text{ and } W(f) = 4\pi \iff f(\Sigma) = S^\rho \]

- For each genus the infimum \(> 4\pi\) is reached: Simon (1993)-Kusner (1996)-Bauer-Kuwert (2003)-Rivi`ere (2010)

- Recent proof of the Willmore Conjecture by Marques-Neves: in genus 1 the minimizer is the Clifford Torus

- Works by Bernard, Bryant, H´elein, Heller, Kilian, Mazzeo, Montiel, Pedit, Pinkall, Ritor´e, Ros, Rosenberg, Sch¨atzle, Schmidt, Schygulla, Topping, Urbano etc.
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In manifolds?

Up to 2010, results just in space forms: Bang-Yen Chen, Guo, Li-Yau, Mazzeo, Montiel, Ritoré, Ros, Urbano, Weiner, etc.

TODAY: prove existence of minimizers or of critical points, in non constantly curved manifolds

Employed techniques:

- Perturbative setting
  - Technique of classical non-linear analysis: Lyapunov-Schmidt reduction
- Global setting
  - Simon's ambient approach (involving GMT: weak objects as varifolds..)
  - Rivière's parametric approach (involving more PDE and functional analysis arguments)
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Perturbative setting

Ambient manifold: \((M, g) = (\mathbb{R}^3, g_\epsilon)\) where \(g_\epsilon^{\mu\nu} := \delta^{\mu\nu} + \epsilon h^{\mu\nu}\), \(h^{\mu\nu}\) is symmetric \((2,0)\) tensor field.

IDEA: for \(\epsilon = 0\) the ambient manifold is \(\mathbb{R}^3 \Rightarrow\) the round spheres form a 4-d manifold of critical points \(\rightarrow\) use a perturbative method lying on a Lyapunov-Schmidt reduction.
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Existence for $W$ in $(\mathbb{R}^3, g_\epsilon)$

NOTATION: if $(M, g) = (\mathbb{R}^3, g_\epsilon := eucl + \epsilon h)$, write $R = \epsilon R_1 + o(\epsilon)$
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Theorem [M.(Math. Zeit. ’10)]
Assume
- $\exists \bar{p} \in \mathbb{R}^3$ such that $R_1(\bar{p}) \neq 0$,
- Said $\| h(p) \| := \sup_{|v|=1} |h_p(v, v)|$
  
  i) $\lim_{|p| \to \infty} \| h(p) \| = 0$.
  
  ii) $\exists C > 0$ and $\alpha > 2$ s.t. $|D_{\lambda} h_{\mu\nu}(p)| < \frac{C}{|p|^{\alpha}} \quad \forall \lambda, \mu, \nu = 1 \ldots 3$. 

Then, for $\epsilon$ small enough, there exists a perturbed standard sphere $S_{\rho \epsilon \bar{p} \epsilon}$ where $\omega_{\epsilon \bar{p} \epsilon} \in \mathcal{C}_4(S^2)$, which is a Willmore embedding of $S^2$ in $(\mathbb{R}^3, g_\epsilon)$.
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\[ S^\rho_\epsilon(p_{\epsilon}, \rho_{\epsilon}) \] (where \(w_{\epsilon}(p_{\epsilon}, \rho_{\epsilon}) \in C^{4,\alpha}(S^2)\)) which is a Willmore embedding of \(S^2\) in \((\mathbb{R}^3, g_\epsilon)\)
Lemma [M. (Math. Z. ’10)]: Let $(M, g)$ be a general ambient manifold with scalar curvature $R$, then the following expansion of $W$ on small geodesic spheres holds:

$$W(S_p, \rho) = 4\pi - \frac{2\pi}{3} R(p) \rho^2 + O(\rho^3)$$

REMARK: $g \in \mathcal{E}$ is close and asymptotic to euclidean but NOT CONSTANT CURVATURE

Related perturbative results, under area constraint, by Lamm-Metzger-Schulze and Lamm-Metzger
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$W_{cnf} = \frac{1}{2} \int |A^o|^2$ in $(\mathbb{S}^3, g_\epsilon)$: introduction
$W_{cnf} = \frac{1}{2} \int |A^\circ|^2$ in $(S^3, g_\epsilon)$: introduction

- By Souam-Toubiana and Manzano-Souam: in Berger spheres (and more generally in non round left invariant metrics on $S^3$) there are NO totally umbilical immersions (i.e. $A^\circ \equiv 0$)
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- Question: Do there exist generalized totally umbilical surfaces? (i.e. critical points of \(W_{conf} := \frac{1}{2} \int |A^\circ|^2 dvol\).)
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- **Fact:** On ANY Riemannian manifold \((M, g)\) by direct computation on shrinking geodesic spheres (M. 2011-JGA)

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\inf_{f:S^2 \hookrightarrow M} W_{conf}(f) = 0.
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\[ W_{cnf} = \frac{1}{2} \int |A^\circ|^2 \text{ in } (S^3, g_\epsilon): \text{ introduction} \]

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\[ \inf_{f : S^2 \hookrightarrow M} W_{conf}(f) = 0. \]

- \( \Rightarrow \) In Berger spheres: a minimizing sequence either converges to a totally umbilical surface (but this does not exist by S-T) or it shrinks to a point \( \Rightarrow \) minimization cannot be performed \( \Rightarrow \) Perturbative approach, saddle type critical points.
Existence of generalized totally umbilic spheres

**Notation** \((\mathbb{S}^3, g_0)\) = 3-sphere with round metric.
Existence of generalized totally umbilic spheres

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More precisely, every Willmore surface we construct is a normal graph over a totally umbilic sphere in \((\mathbb{S}^3, g_0)\) via a smooth function \(w_\varepsilon\) converging to 0 in \(C^{4, \alpha}\) norm as \(\varepsilon \to 0\).
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**Remark.** The critical points we construct are saddle points for \(W_{conf}\). Moreover, a standard *bumpy-metric argument* shows that (in case \((\mathbb{S}^3, g_\varepsilon)\) *does not* have constant sectional curvature) these are generically non-degenerate of index exactly 4.
The Lie group case

**Corollary** (Carlotto-M.’13) Let $g_\varepsilon = g_0 + \varepsilon h$ be a left-invariant metric on $SU(2) \cong S^3$. There exists $\varepsilon \in \mathbb{R}_{>0}$ such that if $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ then for every $p \in S^3$ there exists an embedded critical 2-sphere for the conformal Willmore functional (in metric $g_\varepsilon$) passing through $p$. As a result, under these assumptions the conformal Willmore functional $W_{conf}$ has **uncountably many** distinct critical points.
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Minimization of $E = \frac{1}{2} \int |A|^2$ in compact manifolds: Introduction

▶ Birkhoff (1917): in any closed Riemannian manifold there exist a closed geodesic.

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**Corollary.** If $(M, g)$ is a compact 3-dimensional Riemannian manifold with strictly positive sectional curvature, then there exists a smooth minimizer of $E = \frac{1}{2} \int |A|^2$ among smooth immersed spheres; i.e. there exists a generalized totally geodesic immersion.
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- Condition a) is to prevent the minimizing sequence to shrink to a point (by contradiction via a blow up argument using the Willmore lower bound $E \geq 4\pi$ in $\mathbb{R}^3$).
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- $\implies$ There exists a Radon measure $\mu$ on $M$ such that $\mu_k \to \mu$ up to subsequences
- The sequence may degenerate: $f_k$ may shrink to a point or $\mu$ may be 0; excluded by a blow up procedure using assumption a).
⇒ using assumptions a) and b) we proved that the minimizing sequence is compact and does not degenerate
Sketch of proof-2: Existence of candidate minimizer

- Using assumptions a) and b) we proved that the minimizing sequence is **compact** and does not degenerate.
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- ⇒ \( \mu \) is a candidate minimizer and we have to prove regularity i.e. this measure is associated to a smooth immersion of a sphere
Sketch of proof-3: Regularity

- Take inspiration from [Simon (CAG '93)] and do a partition of $spt\mu$ into good and bad points:

  $\lim_{\rho \to 0} \liminf_{k \to \infty} \int_{f_k(S^2) \cap B(\xi, \rho)} |A| > \epsilon^2$;

  the complementary are the good points.

  Adapting Simon we proved that near the good points $\mu$ is union of $C^{1,\alpha} \cap W^{2,2}$ graphs.

  Using a topological argument involving Gauss-Bonnet theorem + $\inf E < 4\pi$ we excluded the bad points.

  $\Rightarrow$ regularity everywhere: locally $\mu$ is union of $C^{1,\alpha} \cap W^{2,2}$ graphs.

  Globally?

  By a compactness theorem of Breuning $\mu$ is a $C^{1,\alpha} \cap W^{2,2}$ immersion of a sphere.

  Use the equation + bootstrap $\Rightarrow$ smoothness of the immersion.
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Some comments on the approach

Good news
▶ manage to exclude branch points by a topological argument and low energy framework
▶ prove existence-regularity theorems above

Bad news
▶ In order to exclude branch points we use a topological argument heavily depending on the codimension one assumption,
▶ The regularity heavily relies on the minimizing property rather then the criticality (i.e. the Willmore PDE is satisfied) → not suitable for min-max problems.
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ii) There exists at most finitely many points $\{a_1 \cdots a_N\}$ such that for any compact $K \subset \mathbb{S}^2 \setminus \{a_1 \cdots a_N\}$

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Parametric approach 1: possibly branched lipschitz immersions

By Nash, assume that $M^m \subset \mathbb{R}^n$; for any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$

$$W^{k,p}(S^2, M^m) := \left\{ u \in W^{k,p}(S^2, \mathbb{R}^n) \text{ s.t. } u(x) \in M^m \text{ for a.e. } x \in S^2 \right\}$$

A map $\vec{\Phi} \in W^{1,\infty}(S^2, M^m)$ is a possibly branched lipschitz immersion if

i) there exists $C > 1$ such that

$$\forall x \in S^2 \; C^{-1}|d\vec{\Phi}|^2(x) \leq |d\vec{\Phi} \wedge d\vec{\Phi}|(x) \leq |d\vec{\Phi}|^2(x) \quad (1)$$

ii) There exists at most finitely many points $\{a_1 \cdots a_N\}$ such that for any compact $K \subset S^2 \setminus \{a_1 \cdots a_N\}$

$$\text{ess inf}_{x \in K} |d\vec{\Phi}|(x) > 0. \quad (2)$$
For any possibly branched lipschitz immersion we can define almost everywhere the Gauss map

\[ \vec{n}_\Phi := \star h \frac{\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}|} \in \wedge^{m-2} T_{\Phi(x)} M^m \]
For any possibly branched Lipschitz immersion we can define almost everywhere the Gauss map

$$\tilde{n}_\Phi := (*) H \frac{\partial_{x_1} \Phi \wedge \partial_{x_2} \Phi}{|\partial_{x_1} \Phi \wedge \partial_{x_2} \Phi|} \in \wedge^{m-2} T_{\Phi(x)} M^m$$

Definition: [M., Rivière '11] A possibly branched Lipschitz immersion $\Phi \in W^{1,\infty}(S^2, M^m)$ is called "weak, possibly branched, immersion" if the Gauss map satisfies

$$\int_{S^2} |D \tilde{n}_\Phi|^2 dvol_g < +\infty.$$ (3)
For any *possibly branched lipschitz immersion* we can define almost everywhere the *Gauss map*

$$\vec{n}_\Phi := \star_h \frac{\partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}}{|\partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}|} \in \wedge^{m-2} T_{\vec{\Phi}(x)} M^m$$

**Definition:** [M., Rivière ’11] A possibly branched lipschitz immersion $\vec{\Phi} \in W^{1,\infty}(S^2, M^m)$ is called "weak, possibly branched, immersion" if the Gauss map satisfies

$$\int_{S^2} |D\vec{n}_\Phi|^2 \ dvol_g < +\infty. \quad (3)$$

The space of "weak, possibly branched, immersions" of $S^2$ into $M^m$ is denoted $\mathcal{F}_{S^2}$. 
Parametric approach 2: weak, possibly branched, immersions

For any possibly branched lipschitz immersion we can define almost everywhere the Gauss map

\[ \vec{n}_\Phi := *_h \frac{\partial \vec{x}_1 \Phi \wedge \partial \vec{x}_2 \Phi}{|\partial \vec{x}_1 \Phi \wedge \partial \vec{x}_2 \Phi|} \in \wedge^{m-2} T_{\Phi(x)} M^m \]

Definition: [M., Rivièrè ’11] A possibly branched lipschitz immersion \( \vec{\Phi} \in W^{1,\infty}(\mathbb{S}^2, M^m) \) is called ”weak, possibly branched, immersion” if the Gauss map satisfies

\[ \int_{\mathbb{S}^2} |D\vec{n}_\Phi|^2 \, dvol_g < +\infty. \] (3)

The space of "weak, possibly branched, immersions" of \( \mathbb{S}^2 \) into \( M^m \) is denoted \( \mathcal{F}_{\mathbb{S}^2} \).

\( \rightarrow \) right functional space where defining \( W, W_{conf}, E, \ldots \).
**Proposition:** [Toro, Müller-Sverak, Hélein, Rivière]

Let $\vec{\Phi} \in \mathcal{F}_{S^2}$ then $\exists \Psi : S^2 \rightarrow S^2$ bilipschitz homeomorphism such that $\vec{\Phi} \circ \Psi$ is weakly conformal: almost everywhere on $S^2$

\[
\begin{align*}
| \partial_{x_1} (\vec{\Phi} \circ \Psi) |^2_h &= | \partial_{x_2} (\vec{\Phi} \circ \Psi) |^2_h \\
\langle h(\partial_{x_1} (\vec{\Phi} \circ \Psi), \partial_{x_2} (\vec{\Phi} \circ \Psi)) \rangle &= 0
\end{align*}
\]

where $(x_1, x_2)$ are local arbitrary conformal coordinates in $S^2$ for the standard metric. Moreover $\vec{\Phi} \circ \Psi$ is in $W^{2,2} \cap W^{1,\infty}(S^2, M^m)$. 
Proposition: [Toro, Müller-Sverak, Hélein, Rivière]
Let $\Phi \in \mathcal{F}_{S^2}$ then $\exists \Psi : S^2 \rightarrow S^2$ bilipschitz homeomorphism such that $\Phi \circ \Psi$ is weakly conformal: almost everywhere on $S^2$

$$
\begin{cases}
|\partial_{x_1}(\Phi \circ \Psi)|^2_h = |\partial_{x_2}(\Phi \circ \Psi)|^2_h \\
h(\partial_{x_1}(\Phi \circ \Psi), \partial_{x_2}(\Phi \circ \Psi)) = 0
\end{cases}
$$

where $(x_1, x_2)$ are local arbitrary conformal coordinates in $S^2$ for the standard metric. Moreover $\Phi \circ \Psi$ is in $W^{2,2} \cap W^{1,\infty}(S^2, M^m)$.

Remark: We don’t ask conformality from the beginning for variational reasons
Theorem[M., Rivièrè ’11] Let $\Phi_k \in \mathcal{F}_{S^2}$ be conformal such that

$$\limsup_{k \to +\infty} \int_{S^2} \left[ 1 + |D\vec{n}_{\Phi_k}|^2_h \right] dvol_{g_k} < +\infty \quad \liminf_{k \to +\infty} \text{diam}(\Phi_k(S^2)) > 0.$$  

(4)
Parametric approach 4: Relative compactness in $\mathcal{F}_{S^2}$

**Theorem** [M., Rivière ’11] Let $\Phi_k \in \mathcal{F}_{S^2}$ be conformal such that

$$\limsup_{k \to +\infty} \int_{S^2} \left[ 1 + \left| D\vec{n}_{\Phi_k} \right|^2_h \right] \ dvol_{g_k} < +\infty \quad \liminf_{k \to +\infty} \text{diam}(\Phi_k(S^2)) > 0.$$  

(4)

Then, up to subsequences in $k$, $\exists \psi_k : S^2 \to S^2$ bilipschitz homeomorphism,

$$\Phi_k \circ \psi_k \to \vec{f}_\infty \in W^{1,\infty}(S^2, M^m) \quad \text{strongly in } C^0(S^2, M^m).$$  

(5)
Theorem [M., Rivièrem ’11] Let $\Phi_k \in \mathcal{F}_{S^2}$ be conformal such that
\[
\limsup_{k \to +\infty} \int_{S^2} \left[ 1 + |D\vec{n}_{\Phi_k}|^2_h \right] \, dvol_{g_k} < +\infty \quad \text{and} \quad \liminf_{k \to +\infty} \text{diam}(\Phi_k(S^2)) > 0.
\]

(4)

Then, up to subsequences in $k$, $\exists \Psi_k : S^2 \to S^2$ bilipschitz homeomorphism,
\[
\Phi_k \circ \Psi_k \longrightarrow \vec{f}_\infty \in W^{1,\infty}(S^2, M^m) \quad \text{strongly in } C^0(S^2, M^m).
\]

Moreover $\exists (f^i_k)_{i=1}^N \subset \mathcal{M}^+(S^2)$, for every $1 \leq i \leq N$
$\exists b^{i,1} \ldots b^{i,N_i}$ such that
\[
\Phi_k \circ f^i_k \rightharpoonup \vec{\xi}_\infty \text{ weakly in } W^{2,2}_{loc}(S^2 \setminus \{b^{i,1} \ldots b^{i,N_i}\}),
\]
where $\vec{\xi}_\infty \in \mathcal{F}_{S^2}$ is conformal.
**Parametric approach 4: Relative compactness in $\mathcal{F}_{S^2}$**

**Theorem**[M., Rivièr e ’11] Let $\vec{\Phi}_k \in \mathcal{F}_{S^2}$ be conformal such that

$$
\limsup_{k \to +\infty} \int_{S^2} \left[ 1 + |D\vec{n}_{\vec{\Phi}_k}|^2_h \right] \, d\text{vol}_{g_k} < +\infty \quad \liminf_{k \to +\infty} \text{diam}(\vec{\Phi}_k(S^2)) > 0.
$$

(4)

Then, up to subsequences in $k$, $\exists \Psi_k : S^2 \to S^2$ bilipschitz homeomorphism,

$$
\vec{\Phi}_k \circ \Psi_k \longrightarrow \vec{f}_\infty \in W^{1,\infty}(S^2, M^m) \quad \text{strongly in } C^0(S^2, M^m). (5)
$$

Moreover $\exists (f^i_k)_{i=1}^N \subset \mathcal{M}^+(S^2)$, for every $1 \leq i \leq N$

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$$

where $\vec{\xi}_\infty \in \mathcal{F}_{S^2}$ is conformal. In addition we have

$$
\vec{f}_\infty(S^2) = \bigcup_{i=1}^N \vec{\xi}_\infty(S^2), \quad A(\vec{\Phi}_k) \to A(\vec{f}_\infty), \quad (\vec{f}_\infty)_*[S^2] = \sum_{i=1}^N (\vec{\xi}_\infty)_*[S^2].
$$

**Remark:** related compactness, independently, by Chen-Li (2011)
Parametric approach 4: Relative compactness in $\mathcal{F}_S^2$

**Theorem**[M., Rivière ’11] Let $\vec{\Phi}_k \in \mathcal{F}_S^2$ be conformal such that

$$\limsup_{k \to +\infty} \int_{S^2} \left[ 1 + |D\vec{n}_{\Phi_k}|^2 \right] dvol_{g_k} < +\infty \quad \liminf_{k \to +\infty} diam(\vec{\Phi}_k(S^2)) > 0.$$ (4)

Then, up to subsequences in $k$, $\exists \psi_k : S^2 \to S^2$ bilipschitz homeomorphism,

$$\vec{\Phi}_k \circ \psi_k \longrightarrow \vec{f}_\infty \in W^{1,\infty}(S^2, M^m) \quad \text{strongly in } C^0(S^2, M^m).$$ (5)

Moreover $\exists (f^i_k)_{i=1\ldots N} \subset \mathcal{M}^+(S^2)$, for every $1 \leq i \leq N$

$\exists b^{i,1} \ldots b^{i,N_i}$ such that

$$\vec{\Phi}_k \circ f^i_k \rightharpoonup \vec{\xi}_\infty^i \text{ weakly in } W^{2,2}_{loc}(S^2 \setminus \{b^{i,1} \ldots b^{i,N_i}\}),$$

where $\vec{\xi}_\infty^i \in \mathcal{F}_S^2$ is conformal. In addition we have

$$\vec{f}_\infty(S^2) = \bigcup_{i=1}^N \vec{\xi}_\infty^i(S^2), \quad A(\vec{\Phi}_k) \to A(\vec{f}_\infty), \quad (\vec{f}_\infty)_*[S^2] = \sum_{i=1}^N (\vec{\xi}_\infty^i)_*[S^2].$$

**Remark:** related compactness, independently, by Chen-Li (2011)
**Proposition [M., Rivièrè ’12]** Let $\Phi \in \mathcal{F}_{S^2}$, then $W$ is Fréchet differentiable for normal $W^{1,\infty} \cap W^{2,2}$ pertubations supported away from the branched points: $spt(\bar{w}) \subset S^2 \setminus \bigcup_{i=1}^N b^i$. 

**Theorem [M., Rivièrè ’12]**

For all critical points $\vec{\Phi} \in \mathcal{F}_{S^2}$ if $d\vec{\Phi} W = 0$ then $\vec{\Phi}$ is $C^\infty$ outside the finitely many branched points. 

**Remark:** Regularity for all critical points $\rightarrow$ suitable for saddle points.
Proposition [M., Rivière ’12] Let $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$, then $W$ is Fréchet differentiable for normal $W^{1,\infty} \cap W^{2,2}$ pertubations supported away from the branched points: $spt(\vec{w}) \subset \mathbb{S}^2 \setminus \bigcup_{i=1}^{N} b^i$.

Theorem [M., Rivière ’12] $dW_{\vec{\Phi}} = 0$ if and only if

$$
\frac{1}{2} D^{*g}_g \left[ D_g \vec{H} - 3\pi \vec{\eta}(D_g \vec{H}) + \ast_h \left( (\ast_g D_g \vec{\eta}) \wedge M \vec{H} \right) \right] = \tilde{R}(\vec{H}) - R^{\perp}_{\vec{\Phi}}(T\vec{\Phi})
$$

where $\tilde{R}(\vec{X}) := -\pi \vec{\eta} \left[ \sum_{i=1}^{2} Riem^h(\vec{X}, \vec{e}_i) \vec{e}_i \right]$ and $R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}) := \left( \pi_T \left[ Riem^h(\vec{e}_1, \vec{e}_2) \vec{H} \right] \right)^\perp$. 

Remark: Regularity for all critical points $\rightarrow$ suitable for saddle points.
**Proposition** [M., Rivière '12] Let \( \Phi \in \mathcal{F}_{S^2} \), then \( W \) is Fréchet differentiable for normal \( W^{1,\infty} \cap W^{2,2} \) pertubations supported away from the branched points: \( spt(\vec{w}) \subset S^2 \setminus \bigcup_{i=1}^{N} b_i \).

**Theorem** [M., Rivière '12] \( dW_{\Phi} = 0 \) if and only if

\[
\frac{1}{2} D_{g}^{*} \left[ D_{g} \vec{H} - 3\pi_{\vec{n}}(D_{g} \vec{H}) + *_{h} \left( (*_{g} D_{g} \vec{n}) \wedge_{M} \vec{H} \right) \right] = \tilde{R}(\vec{H}) - R_{\Phi}^{\perp}(T \vec{\Phi})
\]

where \( \tilde{R}(\vec{X}) := -\pi_{\vec{n}} \left[ \sum_{i=1}^{2} Riem^{h}(\vec{X}, \vec{e}_{i})\vec{e}_{i} \right] \) and

\[
R_{\Phi}^{\perp}(T \vec{\Phi}) := \left( \pi_{T} \left[ Riem^{h}(\vec{e}_{1}, \vec{e}_{2})\vec{H} \right] \right)^{\perp}.
\]

**Theorem** [M., Rivière '12] \( \forall \Phi \in \mathcal{F}_{S^2} \) if \( d_{\Phi}W = 0 \) then \( \Phi \) is \( C^{\infty} \) outside the finitely many branched points.
Parametric approach 5: Regularity

**Proposition** [M., Rivière ’12] Let $\Phi \in \mathcal{F}_{S^2}$, then $W$ is Fréchet differentiable for normal $W^{1,\infty} \cap W^{2,2}$ pertubations supported away from the branched points: $\text{spt}(\vec{w}) \subset S^2 \setminus \bigcup_{i=1}^{N} b^i$.

**Theorem** [M., Rivière ’12] $dW_{\vec{\Phi}} = 0$ if and only if

$$\frac{1}{2} D^*_{g} \left[ D_{g} \vec{H} - 3\pi_{\vec{n}}(D_{g} \vec{H}) + \star_h \left( (\star_{g} D_{g} \vec{n}) \wedge_{M} \vec{H} \right) \right] = \tilde{R}(\vec{H}) - R_{\vec{\Phi}}^{\perp}(T\vec{\Phi})$$

where $\tilde{R}(\vec{X}) := -\pi_{\vec{n}} \left[ \sum_{i=1}^{2} \text{Riem}^h(\vec{X}, \vec{e}_i)\vec{e}_i \right]$ and $R_{\vec{\Phi}}^{\perp}(T\vec{\Phi}) := \left( \pi_T \left[ \text{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{H} \right] \right)^\perp$.

**Theorem** [M., Rivière ’12] $\forall \vec{\Phi} \in \mathcal{F}_{S^2}$ if $d_{\vec{\Phi}}W = 0$ then $\vec{\Phi}$ is $C^\infty$ outside the finitely many branched points.

**Remark:** Regularity for all critical points $\to$ suitable for saddle points.
Application: Willmore spheres in homotopy classes

**Theorem** [M., Rivière '12] Fix $0 \neq \gamma \in \pi_2(M^m)$
Application: Willmore spheres in homotopy classes

**Theorem** [M., Rivièrè ’12] Fix $0 \neq \gamma \in \pi_2(M^m)$. Then there exist finitely many branched conformal immersions $\Phi^1, \ldots, \Phi^N \in \mathcal{F}_{S^2}$ and a Lipschitz map $\vec{f} \in W^{1,\infty}(S^2, M^m)$ with $[\vec{f}] = \gamma$ satisfying

$$
\vec{f}(S^2) = \bigcup_{i=1}^{N} \Phi^i(S^2), \quad \vec{f}_*[S^2] = \sum_{i=1}^{N} \Phi^i_*[S^2].
$$

Remark: the Theorem completes the result of Sacks-Uhlembeck about area minimizing branched spheres in homotopy groups; they prove that, if $\pi_1(M^m) = 0$ then there exists area minimizing (in their $\pi_0(C^0(S^2, M^m))$ class) branched immersions generating $\pi_2(M^m)$, but since bubbling can occur, it is not clear which are the 2-homotopy classes having an area minimizing representant.
Application: Willmore spheres in homotopy classes

**Theorem** [M., Riviè ` re ’12] Fix $0 \neq \gamma \in \pi_2(M^m)$. Then there exist finitely many branched conformal immersions $\vec{\Phi}^1, \ldots, \vec{\Phi}^N \in \mathcal{F}_{\mathbb{S}^2}$ and a Lipschitz map $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ with $[\vec{f}] = \gamma$ satisfying

$$\vec{f}(\mathbb{S}^2) = \bigcup_{i=1}^{N} \vec{\Phi}^i(\mathbb{S}^2), \quad \vec{f}_*[\mathbb{S}^2] = \sum_{i=1}^{N} \vec{\Phi}^i_*[\mathbb{S}^2].$$

Moreover for every $i$, the map $\vec{\Phi}_i$ is a conformal branched area-constrained Willmore immersion which is smooth outside the finitely many branched points $b^1, \ldots, b^{N_i}$. 
Theorem [M., Rivièr ’12] Fix $0 \neq \gamma \in \pi_2(M^m)$. Then there exist finitely many branched conformal immersions $\Phi^1, \ldots, \Phi^N \in \mathcal{F}_{S^2}$ and a Lipschitz map $\vec{f} \in W^{1,\infty}(S^2, M^m)$ with $[\vec{f}] = \gamma$ satisfying

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Application: Willmore spheres in homotopy classes

**Theorem** [M., Rivièrè '12] Fix $0 \neq \gamma \in \pi_2(M^m)$. Then there exist finitely many branched conformal immersions $\Phi^1, \ldots, \Phi^N \in \mathcal{F}_{\mathbb{S}^2}$ and a Lipschitz map $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ with $[\vec{f}] = \gamma$ satisfying

$$\vec{f}(\mathbb{S}^2) = \bigcup_{i=1}^{N} \Phi^i(\mathbb{S}^2), \quad \vec{f}_*[\mathbb{S}^2] = \sum_{i=1}^{N} \Phi^i_*[\mathbb{S}^2].$$

Moreover for every $i$, the map $\Phi^i$ is a conformal branched area-constrained Willmore immersion which is smooth outside the finitely many branched points $b^1, \ldots, b^{N_i}$.

**Remark:** the Theorem completes the result of Sacks-Uhlembeck about area minimizing branched spheres in homotopy groups; they prove that, if $\pi_1(M^m) = 0$ then there exists area minimizing (in their $\pi_0(C^0(\mathbb{S}^2, M))$ class) branched immersions generating $\pi_2(M^m)$, but since bubbling can occur, it is not clear which are the 2-homotopy classes having an area minimizing representant.
Theorem [M., Rivièrè ’12] Fix \( 0 \neq \gamma \in \pi_2(M^m) \). Then there exist finitely many branched conformal immersions \( \Phi^1, \ldots, \Phi^N \in \mathcal{F}_{\mathbb{S}^2} \) and a Lipschitz map \( \vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m) \) with \( [\vec{f}] = \gamma \) satisfying

\[
\vec{f}(\mathbb{S}^2) = \bigcup_{i=1}^N \Phi^i(\mathbb{S}^2), \quad \vec{f}_*[\mathbb{S}^2] = \sum_{i=1}^N \Phi^i_*[\mathbb{S}^2].
\]

Moreover for every \( i \), the map \( \Phi^i \) is a conformal branched area-constrained Willmore immersion which is smooth outside the finitely many branched points \( b^1, \ldots, b^{Ni} \).

Remark: the Theorem completes the result of Sacks-Uhlembeck about area minimizing branched spheres in homotopy groups; they prove that, if \( \pi_1(M^m) = 0 \) then there exists area minimizing (in their \( \pi_0(C^0(\mathbb{S}^2, M)) \) class) branched immersions generating \( \pi_2(M^m) \), but since bubbling can occur, it is not clear which are the 2-homotopy classes having an area minimizing representant. → minimize \( \text{Area} + W \).
Theorem [M., Rivièvre ’12]
Let $(M^m, h)$ be a compact Riemannian manifold and fix any $A > 0$. Then there exist finitely many branched conformal immersions $\vec{\Phi}_1, \ldots, \vec{\Phi}_N \in \mathbb{F}_2$ and a Lipschitz map $\vec{f} \in W^{1, \infty}(S^2, M^m)$ with
\[ \sum_{i=1}^{N} A(\vec{\Phi}_i) = A, \quad \vec{f}(S^2) = \bigcup_{i=1}^{N} \vec{\Phi}_i(S^2), \quad \vec{f}^* [S^2] = \sum_{i=1}^{N} \vec{\Phi}_i^* [S^2], \]
such that for every $i$, the map $\vec{\Phi}_i$ is a conformal branched area-constraint Willmore immersion which is smooth outside the finitely many branched points $b_1, \ldots, b_N$.

Remark: the theorem extends to arbitrary area the analogous perturbative result of Lamm-Metzger proved for infinitesimal area constraint (for small area there is just one sphere).
Theorem [M., Riviè `ere ’12]
Let \((M^m, h)\) be a compact Riemannian manifold and fix any \(A > 0\). Then there exist finitely many branched conformal immersions \(\Phi^1, \ldots, \Phi^N \in \mathcal{F}_{S^2}\) and a Lipschitz map \(\tilde{f} \in W^{1,\infty}(S^2, M^m)\) with

\[
\sum_{i=1}^{N} A(\Phi^i) = A, \quad \tilde{f}(S^2) = \bigcup_{i=1}^{N} \Phi^i(S^2), \quad \tilde{f}_*[S^2] = \sum_{i=1}^{N} \Phi^i_*[S^2],
\]

such that for every \(i\), the map \(\Phi^i\) is a conformal branched area-constraint Willmore immersion which is smooth outside the finitely many branched points \(b^1, \ldots, b^{N_i}\).
Application 2: Willmore spheres under area constraint

**Theorem**[M., Rivière ’12]

Let \((M^m, h)\) be a compact Riemannian manifold and fix any \(A > 0\). Then there exist finitely many branched conformal immersions \(\vec{\Phi}^1, \ldots, \vec{\Phi}^N \in \mathcal{F}_{S^2}\) and a Lipschitz map \(\vec{f} \in W^{1,\infty}(S^2, M^m)\) with

\[
\sum_{i=1}^{N} A(\vec{\Phi}^i) = A, \quad \vec{f}(S^2) = \bigcup_{i=1}^{N} \vec{\Phi}^i(S^2), \quad \vec{f}_*[S^2] = \sum_{i=1}^{N} \vec{\Phi}^i_*[S^2],
\]

such that for every \(i\), the map \(\vec{\Phi}^i\) is a conformal branched area-constraint Willmore immersion which is smooth outside the finitely many branched points \(b^1, \ldots, b^{N_i}\).

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Theorem [M., Rivièr e '12]
Let \((M^m, h)\) be a compact Riemannian manifold and fix any \(A > 0\). Then there exist finitely many branched conformal immersions \(\bar{\Phi}^1, \ldots, \bar{\Phi}^N \in \mathcal{F}_{S^2}\) and a Lipschitz map \(\bar{f} \in W^{1,\infty}(S^2, M^m)\) with

\[
\sum_{i=1}^{N} A(\bar{\Phi}^i) = A, \quad \bar{f}(S^2) = \bigcup_{i=1}^{N} \bar{\Phi}^i(S^2), \quad \bar{f}_*[S^2] = \sum_{i=1}^{N} \Phi_*^i[S^2],
\]

such that for every \(i\), the map \(\bar{\Phi}^i\) is a conformal branched area-constraint Willmore immersion which is smooth outside the finitely many branched points \(b^1, \ldots, b^{N_i}\).

Remark: the theorem extends to arbitrary area the analogous perturbative result of Lamm-Metzger proved for infinitesimal area constraint (for small area there is just one sphere)
Let \((M^3, g)\) be a Berger sphere with positive sectional curvature (or more generally a left invariant metric on \(S^3\) with positive sectional curvature) and let \(\Sigma\) be minimizer of \(\int |A|^2\) among smooth immersed 2-spheres.

- Has \(\Sigma\) some symmetry (e.g. rotational in Berger)?
- Is the minimizer unique (up to isometries)?
- Who is \(\Sigma\)?

In the minimization of \(\int 1 + |A|^2\) or \(\int 1 + H^2\) in arbitrary codimension is it really convenient that the minimizer splits in a chain of spheres rather than having just one sphere? - Under which conditions (e.g. curvature bounds in the ambient manifold) just one sphere is better?
Let \((M^3, g)\) be a Berger sphere with positive sectional curvature (or more generally a left invariant metric on \(S^3\) with positive sectional curvature) and let \(\Sigma\) be minimizer of \(\int |A|^2\) among smooth immersed 2-spheres.

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Let \((M^3, g)\) be a Berger sphere with positive sectional curvature (or more generally a left invariant metric on \(S^3\) with positive sectional curvature) and let \(\Sigma\) be minimizer of \(\int |A|^2\) among smooth immersed 2-spheres.

a) Has \(\Sigma\) some symmetry (e.g. rotational in Berger)?

b) Is the minimizer unique (up to isometries)?
Some questions

- Let \((M^3, g)\) be a Berger sphere with positive sectional curvature (or more generally a left invariant metric on \(S^3\) with positive sectional curvature) and let \(\Sigma\) be minimizer of \(\int |A|^2\) among smooth immersed 2-spheres.
  
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Let \((M^3, g)\) be a Berger sphere with positive sectional curvature (or more generally a left invariant metric on \(S^3\) with positive sectional curvature) and let \(\Sigma\) be minimizer of \(\int |A|^2\) among smooth immersed 2-spheres.

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In the minimization of \(\int 1 + |A|^2\) or \(\int 1 + H^2\) in arbitrary codimension is it really convenient that the minimizer splits in a chain of spheres rather than having just one sphere?
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Let \((M^3, g)\) be a Berger sphere with positive sectional curvature (or more generally a left invariant metric on \(S^3\) with positive sectional curvature) and let \(\Sigma\) be minimizer of \(\int |A|^2\) among smooth immersed 2-spheres.

a) Has \(\Sigma\) some symmetry (e.g. rotational in Berger)?

b) Is the minimizer unique (up to isometries)?

b) Who is \(\Sigma\)?

In the minimization of \(\int 1 + |A|^2\) or \(\int 1 + H^2\) in arbitrary codimension is it really convenient that the minimizer splits in a chain of spheres rather than having just one sphere?

- Under which conditions (e.g. curvature bounds in the ambient manifold) just one sphere is better?
The articles

!!THANK YOU FOR THE ATTENTION!!