Nonproper Minimal Surfaces with Arbitrary Topology in $H^3$

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20 June 2013
Basic Definitions

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- A compact, orientable surface with boundary is called **absolutely area minimizing surface** if it has the smallest area among all orientable surfaces (with no topological restriction) with the same boundary.

  A noncompact, orientable surface is called **absolutely area minimizing surface** if any compact subsurface is an absolutely area minimizing surface.
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Any least area disk, and area minimizing surface is automatically a minimal surface. The main difference between least area disk and area minimizing surface is that there is no topological restriction on the surface.
Calabi-Yau Conjecture in $\mathbb{R}^3$

A complete, embedded minimal surface in $\mathbb{R}^3$ is proper.
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- **Finite Genus case:** Finite genus & uncountable number of ends case is still open.

- **Constant Mean Curvature case:** [Meeks-Tinaglia] The conjecture is true for $H$-surfaces in $\mathbb{R}^3$. 

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Nonproper Minimal Surfaces with Arbitrary Topology in $\mathbb{H}^3$
If $\Sigma$ is a complete, embedded minimal surface in $H^3$, then does $\Sigma$ necessarily be properly embedded, like in $R^3$ case? The answer is No. There exists a complete, nonproper, minimal plane in $H^3$. [C–2011] Question Are there other complete nonproper, minimal surfaces in $H^3$?
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**Question**

Are there other complete nonproper, minimal surfaces in $H^3$?
What type of surfaces can be minimally and completely embedded in $H^3$?

**Finite Topology:**

[Oliviera-Soret-1998] If $S$ has finite genus and finite number of ends, then there exists a complete, proper minimal surface $\Sigma$ in $H^3$ with $\Sigma \cong S$.

**Arbitrary Topology:**

[Martin-White-2012] For any $S$, there exists complete, proper area minimizing surface $\Sigma$ in $H^3$ with $\Sigma \cong S$.

What type of surfaces can be nonproperly embedded in $H^3$ as a complete minimal surface?
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What type of surfaces can be minimally and completely embedded in $\mathbb{H}^3$?

Finite Topology: [Oliviera-Soret-1998] If $S$ has finite genus and finite number of ends, then there exists a complete, proper minimal surface $\Sigma$ in $\mathbb{H}^3$ with $\Sigma \simeq S$. 

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Topography of the Complete Minimal Surfaces in $H^3$

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- **Finite Topology:** [Oliviera-Soret-1998] If $S$ has finite genus and finite number of ends, then there exists a complete, proper minimal surface $\Sigma$ in $H^3$ with $\Sigma \approx S$.

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What type of surfaces can be nonproperly embedded in $H^3$ as a complete minimal surface?
Main Result:

Theorem:

Any open, orientable surface $S$ can be **nonproperly** embedded in $H^3$ as a complete minimal surface.
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Let $S$ be given.

- Let $\Sigma_1$ be a complete, minimal surface in $H^3$ with $\Sigma_1 \sim S$ [MW]
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- Let $\Sigma_1$ be a complete, minimal surface in $H^3$ with $\Sigma_1 \sim S$  [MW]
- Let $\Sigma_2$ be the nonproper minimal plane in $H^3$.  [C–]
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- $\Sigma$ is both nonproper and $\Sigma \sim S$. 

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Nonproper Minimal Surfaces with Arbitrary Topology in $H^3$
Step 1: Nonproper Minimal Plane in $H^3$

Outline:

- Take sequence of circles $C_n$ in $S^2_{\infty}(H^3)$ limiting on equator.
- Each $C_n$ bounds a geodesic plane $P_n$ in $H^3$.
- Connect $P_n$ and $P_{n+1}$ with a bridge at infinity (alternating sides).
- Resulting plane $\Sigma_1$ is nonproperly embedded.

The construction is not trivial since we do not have the bridge principle at infinity in $H^3$ for stable minimal surfaces.
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The construction is not trivial since we do not have the bridge principle at infinity in $H^3$ for stable minimal surfaces.
Here, adding a bridge to the same boundary component of a surface would correspond to the pair of pants case. Adding two bridges successively to the same boundary component would correspond to the cylinder with a handle case. In particular, if $C$ is the boundary component in $\partial S_n$ and the annulus $A$ is a small neighborhood of $C$ in $S_n$, then $A \cup B_n$ would be a pair of pants, where $B_n$ is the bridge attached to $C$. On the other hand, if $B'_n$ is a smaller bridge connecting the different sides of the bridge $B_n$, let $B_n \cup B'_n$ be the handle $H_n$. Then $A \cup H_n$ would be a cylinder with a handle (See Figure 4).

Notice that by attaching a bridge $B_n$, we increase the number of boundary components of $S_n$ by 1 and decrease the euler characteristic by 1, i.e.

![Figure 3](image_url)

**Figure 3.** In the simple exhaustion of $S$, $S_1$ is a disk, and $S_{n+1} - S_n$ contains a unique nonannular part, which is a pair of pants (e.g. $S_4 - S_3$), or a cylinder with a handle (e.g. $S_3 - S_2$).
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The construction is not trivial since we do not have the bridge principle at infinity in $\mathbb{H}^3$ for stable minimal surfaces.
Notice that by lemma 2.7, $S_t \cap S_s = \emptyset$ for $t \neq s$, and hence $V_t \cap V_s = \emptyset$ for $t \neq s$.

Now, consider a short arc segment $\eta$ in $\mathbb{H}^3$ with one endpoint is in $S_{t_1}$ and the other end point is in $S_{t_2}$ where $0 < t_1 < t_2 < \epsilon'$. Hence, $\eta$ intersects all minimizing $H$-surfaces $S_t$ with $\partial_\infty S_t = \Gamma_t$ where $t_1 \leq t \leq t_2$. Now for $t_1 < s < t_2$, define the thickness $\lambda_s$ of $V_s$ as $\lambda_s = |\eta \cap V_s|$, i.e. $\lambda_s$ is the length of the piece of $\eta$ in $V_s$. Hence, if $\Gamma_s$ bounds more than one $H$-surface, then the thickness is not 0. In other words, if $\lambda_s = 0$, then $\Gamma_s$ bounds a unique $H$-surface.

As $V_t \cap V_s = \emptyset$ for $t \neq s$, $\sum_{t_1}^{t_2} \lambda_s < |\eta|$. Hence, for only countably many $s \in [t_1, t_2]$, $\lambda_s > 0$. This implies for all but countably many $s \in [t_1, t_2]$, $\lambda_s = 0$, and hence $\Gamma_s$ bounds a unique minimizing $H$-surface. Similarly, this implies for all but countably many $s \in [0, \epsilon']$, $\Gamma_s$ bounds a unique $H$-surface. The proof follows.

Step 1 and Step 2 implies the existence of a nearby $(0 < t < \epsilon')$ smooth curve $\Gamma_t$ to $\Gamma \cup \alpha$ where $\Gamma_t$ bounds a unique minimizing $H$-surface $S_t$, and $S_t$ has the desired topology, i.e. $S_t \simeq S \cup \tilde{N}_\epsilon(\alpha)$.

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**Figure 3.** In the simple exhaustion of $S$, $S_1$ is a disk, and $S_{n+1} - S_n$ contains a unique nonannular part, which is a pair of pants (e.g. $S_4 - S_3$), or a cylinder with a handle (e.g. $S_3 - S_2$).
\( \Pi_1 \) is the least area plane in \( Y_1 = \mathbb{H}^3 - \Sigma_1 \) where \( \partial_\infty \Pi_1 = \lambda_1 \). In particular, \( \Pi_1 = P_1^{-\#_1 \alpha_1^+} P_2^+ \) and \( \lambda_1 = \gamma_1^{-\#_1 \alpha_1^+} \gamma_2^+ \).

Similarly, one can iterate this process by using appropriate isometry \( \phi_n \) such that \( \lambda_n = \phi_n(\lambda_1) = \gamma_n^{-\#_n \alpha_n^+} \gamma_{n+1}^+ \) is a simple closed curve in the region between \( \gamma_n \) and \( \gamma_{n+1} \). Here, \( \alpha_{2n-1}^+ \) is a line segment in the line \( x = -C \) with endpoints \((-C, r_{2n-1}^-, 0) \) and \((-C, r_{2n}^-, 0) \), while \( \alpha_{2n}^+ \) is a line segment in the line \( x = -C \) with endpoints \((-C, -r_{2n}^-, 0) \) and \((-C, -r_{2n+1}^+, 0) \). In particular, the bridges \( \alpha_{2n-1}^+ \) and \( \alpha_{2n}^+ \) are alternating sides (See Figure 7). Then, let \( \Pi_n = \phi_n(\Pi_1) \) and \( \mathcal{R}_n = \phi_n(\mathcal{R}_1) \). Hence, define \( X_{n+1} = X_n - \mathcal{R}_n \). Notice that \( X_n \) is a mean convex subspace of \( \mathbb{H}^3 \).

Other than being mean convex, we will require one more property on \( X_2 \). By the construction of the least area plane \( \Pi_1 \sim P_1^{-\#_1 \alpha_1^+} P_2^+ \), for smaller choice of \( \rho \), we get a thinner bridge in \( \Pi_1 \) connecting \( P_1 \) and \( P_2 \). In particular, if \( \lambda^m_1 = \gamma_1^{-\#_1 \alpha_1^+} \gamma_2^+ \) is the simple closed curve obtained by connecting \( \gamma_1 \) and \( \gamma_2^+ \) along a bridge along \( \alpha_1^+ \) with thickness \( \rho_m \searrow 0 \), then let \( \Pi^m_1 \) be the least area plane in \( Y_1 \) with \( \partial_\infty \Pi^m_1 = \lambda^m_1 \). By the construction, \( \Pi^m_1 \to P_1^- \cup P_2^+ \) as \( n \to \infty \).
Step 2: Minimal Surfaces of Desired Topology in $\mathbb{H}^3$

- **[Martin-White]** Outline: Let $S$ be given.

   - Start with a simple exhaustion of $S$.
     
     $S = \bigcup_{n=1}^{\infty} S_n$
     
     $S_1 \subset S_2 \subset \ldots \subset S_n \subset \ldots$
     
     $S_n + 1 - S_n$ contains either a pair of pants or a cylinder with handle.

   - Bridge principle at infinity for uniquely minimizing surfaces in $\mathbb{H}^3$.

   - Let $\hat{S}_1$ be a geodesic plane in $\mathbb{H}^3$.
     
     Define the area minimizing surface $\hat{S}_n$ in $\mathbb{H}^3$ with $\hat{S}_n \cong S_n$ inductively:
     
     $\hat{S}_{n+1} = \hat{S}_n \# B_n$
     
     where $B_n$ is either one bridge or two successive bridges.

   - $\Sigma_2 = \lim \hat{S}_n$ is an area minimizing surface in $\mathbb{H}^3$ with $\Sigma_2 \cong S$. 

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Nonproper Minimal Surfaces with Arbitrary Topology in $\mathbb{H}^3$
Step 2: Minimal Surfaces of Desired Topology in $\mathbb{H}^3$

- **[Martin-White]** Outline: Let $S$ be given.
  - Start with a simple exhaustion of $S$ [FMM].
  - i.e. $S = \bigcup_{n=1}^{\infty} S_n$ where $S_1 \subset S_2 \subset \ldots \subset S_n \subset \ldots$ $S_{n+1} - S_n$ contains either *pair of pants* or *cylinder with handle*.
Here, adding a bridge to the same boundary component of a surface would correspond to the pair of pants case. Adding two bridges successively to the same boundary component would correspond to the cylinder with a handle case. In particular, if \( C \) is the boundary component in \( \partial S_n \) and the annulus \( A \) is a small neighborhood of \( C \) in \( S_n \), then \( A \cup B_n \) would be a pair of pants, where \( B_n \) is the bridge attached to \( C \). On the other hand, if \( B'_n \) is a smaller bridge connecting the different sides of the bridge \( B_n \), let \( B_n \cup B'_n \) be the handle \( H_n \). Then \( A \cup H_n \) would be a cylinder with a handle (See Figure 4).

Notice that by attaching a bridge \( B_n \), we increase the number of boundary components of \( S_n \) by 1 and decrease the euler characteristic by 1, i.e. \( \#(\partial S_{n+1}) = \#(\partial S_n) + 1 \) and \( \chi(S_{n+1}) = \chi(S_n) - 1 \). Hence, \( g(S_n) = g(S_{n+1}) \) where \( g(\cdot) \) represents the genus of the surface. Similarly by attaching a handle \( H_n \) to \( S_n \), we keep the number of boundary components same, but decrease the euler characteristic by 2, i.e. \( \#(\partial S_{n+1}) = \#(\partial S_n) \) and \( \chi(S_{n+1}) = \chi(S_n) - 2 \). This implies \( g(S_{n+1}) = g(S_n) + 1 \) with the same number of boundary components.

**Figure 3.** In the simple exhaustion of \( S \), \( S_1 \) is a disk, and \( S_{n+1} - S_n \) contains a unique nonannular part, which is a pair of pants (e.g. \( S_4 - S_3 \)), or a cylinder with a handle (e.g. \( S_3 - S_2 \)).

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- Bridge principle at infinity for *uniquely minimizing surfaces* in $H^3$. 

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◊ Bridge principle at infinity for uniquely minimizing surfaces in $H^3$.

◊ Let $\hat{S}_1$ be a geodesic plane in $H^3$.

Define the area minimizing surface $\hat{S}_n$ in $H^3$ with $\hat{S}_n \sim S_n$ inductively:
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Define the area minimizing surface $\hat{S}_n$ in $\mathbb{H}^3$ with $\hat{S}_n \simeq S_n$ inductively:

- $\hat{S}_{n+1} = \hat{S}_n \# B_n$ where $B_n$ is either one bridge or two successive bridges.
want. Hence, in the Poincare ball model, we can get an increasing sequence \( r_n \rightarrow \infty \) such that \( B_{r_n}(0) \cap \Sigma_{n+1} \simeq S_n \) and \( B_{r_{n+1}}(0) \cap \Sigma_{n+1} \simeq S_{n+1} \).

Now, assume that \( S_{n+1} - S_n \) contains a cylinder with a handle. Again, let \( \gamma \) be the component of \( \partial S_n \) where the cylinder with handle attached, and let \( \gamma' \subset S^2_\infty(\mathbb{H}^3) \) be the corresponding component in \( \partial_\infty \Sigma_n \). Let \( D \) be the disk in \( S^2_\infty(\mathbb{H}^3) \) with \( \partial D = \gamma' \) and \( D \cap \Gamma_n = \gamma' \). Like before, let \( \beta_n \) be a smooth arc segment in \( D \) with \( \beta_n \cap \Gamma_n = \partial \beta_n \subset \gamma' \), and \( \beta_n \perp \gamma' \). Now, by Theorem 3.1 we get a uniquely minimizing \( H \)-surface \( \Sigma'_{n+1} \). Again, by choosing the bridge sufficiently thin, we can make sure that \( B_{r_n} \cap \Sigma'_{n+1} \simeq S_n \). Now, let \( \beta'_n \) be the small smooth arc connecting the opposite sides of the bridge along \( \beta_n \). Similarly, by using Theorem 3.1 we add another tiny bridge along \( \beta'_n \) to \( \Sigma'_{n+1} \) and get a uniquely minimizing \( H \) surface \( \Sigma_{n+1} \) where \( \Sigma_{n+1} \simeq S_{n+1} \). Like before, we can find sufficiently large \( r_{n+1} > r_n \) with \( B_{r_n}(0) \cap \Sigma_{n+1} \simeq S_n \) and \( B_{r_{n+1}}(0) \cap \Sigma_{n+1} \simeq S_{n+1} \).

**Figure 4.** If \( S_{n+1} - S_n \) contains a pair of pants in the simple exhaustion, we add a bridge \( B_n \) so that \( S_n \cup B_n \simeq S_{n+1} \) (left). If \( S_{n+1} - S_n \) contains a cylinder with a handle, then we add a handle \( \mathcal{H}_n \) so that \( S_n \cup \mathcal{H}_n \simeq S_{n+1} \). Here the handle \( \mathcal{H}_n \) is just successive two bridges, i.e. \( \mathcal{H}_n = B_n \cup B'_n \) (right).
♯(∂S_{n+1}) = ♯(∂S_n) + 1 and \(χ(S_{n+1}) = χ(S_n) - 1\). Hence, \(g(S_n) = g(S_{n+1})\) where \(g(.)\) represents the genus of the surface. Similarly by attaching a handle \(H_n\) to \(S_n\), we keep the number of boundary components same, but decrease the euler characteristic by 2, i.e. \(♯(∂S_{n+1}) = ♯(∂S_n)\) and \(χ(S_{n+1}) = χ(S_n) - 2\). This implies \(g(S_{n+1}) = g(S_n) + 1\) with the same number of boundary components.

We start the construction with a minimizing \(H\)-plane \(Σ_1\) (a spherical cap) in \(H^3\) bounding a round circle \(Γ_1\) in \(S^2_\infty(H^3)\). Hence, \(Σ_1 \simeq S_1\). Now, we continue inductively (See Figure 5). Assume that \(S_{n+1} - S_n\) contains a pair of pants. Let the pair of pants attached to the component \(γ\) in \(∂S_n\). Let \(γ'\) be the corresponding component of \(Γ_n = ∂_∞Σ_n\). By construction, \(γ'\) bounds a disk \(D\) in \(S^2_\infty(H^3)\) with \(D \cap Γ_n = γ'\). Let \(β_n\) be a smooth arc segment in \(D\) with \(β_n \cap Γ_n = ∂β_n \subset γ'\), and \(β_n \perp γ'\). Now, as \(Σ_n\) is uniquely minimizing \(H\)-surface, and \(β_n\) satisfies the conditions by using the Theorem 3.1, we get a uniquely minimizing \(H\)-surface \(Σ'_{n+1}\). Again, by choosing the bridge along \(β_n\) as thin as we want. Hence, in the Poincare ball model, we can get an increasing sequence \(r_n \to \infty\) such that \(B_{r_n}(0) \cap Σ_{n+1} \simeq S_n\) and \(B_{r_{n+1}}(0) \cap Σ_{n+1} \simeq S_{n+1}\).

Now, assume that \(S_{n+1} - S_n\) contains a cylinder with a handle. Again, let \(γ\) be the component of \(∂S_n\) where the cylinder with handle attached, and let \(γ' \subset S^2_\infty(H^3)\) be the corresponding component in \(∂_∞Σ_n\). Let \(D\) be the disk in \(S^2_\infty(H^3)\) with \(∂D = γ'\) and \(D \cap Γ_n = γ'\). Like before, let \(β_n\) be a smooth arc segment in \(D\) with \(β_n \cap Γ_n = ∂β_n \subset γ'\), and \(β_n \perp γ'\). Now, by Theorem 3.1 we get a uniquely minimizing \(H\)-surface \(Σ'_{n+1}\). Again, by choosing the

\[\text{Figure 5. } Σ_1 \text{ is a uniquely minimizing } H\text{-surface where } ∂_∞Σ_1 \text{ is a round circle. If } S_2 - S_1 \text{ contains a pair of pants, we attach one bridge } B_1 \text{ along } β_1 \text{ to } Σ_1, \text{ and get } Σ_2 = Σ_1♯B_1 \text{ (left). If } S_2 - S_1 \text{ contains a cylinder with a handle, we attach two bridges successively to } Σ_1 \text{ and get } Σ_2 = Σ_1♯H_1 \text{ (right).} \]
Step 2: Minimal Surfaces of Desired Topology in $\mathbb{H}^3$

- **[Martin-White]** Outline: Let $S$ be given.
  - Start with a simple exhaustion of $S$ [FMM].
    - $S = \bigcup_{n=1}^{\infty} S_n$ where $S_1 \subset S_2 \subset \ldots \subset S_n \subset \ldots$
    - $S_{n+1} - S_n$ contains either *pair of pants* or *cylinder with handle*.
  - Bridge principle at infinity for **uniquely minimizing surfaces** in $\mathbb{H}^3$.
  - Let $\hat{S}_1$ be a geodesic plane in $\mathbb{H}^3$.
  - Define the area minimizing surface $\hat{S}_n$ in $\mathbb{H}^3$ with $\hat{S}_n \simeq S_n$ inductively:
    - $\hat{S}_{n+1} = \hat{S}_n \# B_n$ where $B_n$ is either one bridge or two successive bridges.
  - $\Sigma_2 = \lim \hat{S}_n$ is an area minimizing surface in $\mathbb{H}^3$ with $\Sigma_2 \simeq S$. 
Step 3: The Sequence

- Define a sequence of minimal surfaces $\{T_n\}$ inductively.
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- Define a sequence of minimal surfaces \( \{T_n\} \) inductively.

- \( T_1 = \hat{S}_1 \) and \( T_2 = \hat{S}_1 \# \mu P_1 \). Let \( \partial_\infty T_n = \Gamma_n \).
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- \( T_{2n+1} = T_{2n} \# \mathcal{B}_n \) \( (T_{2n} \text{ uniquely minimizing}) \)
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**PROBLEM**

\( T_{2n-1} \cup P_n \) may not be area minimizing in \( H^3 \).
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**PROBLEM**

\( T_{2n-1} \cup P_n \) may not be area minimizing in \( H^3 \).

**NEED**

Mean Convex Subspaces \( X_n \) in \( H^3 \) where \( T_{2n-1} \cup P_n \) is uniquely minimizing in \( X_n \).
Step 4: Mean Convex Subspaces $X_n$ in $\mathbb{H}^3$

- We want $T_{2n-1} \cup P_n$ to be uniquely minimizing in $X_n \subset \mathbb{H}^3$. 

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Nonproper Minimal Surfaces with Arbitrary Topology in $\mathbb{H}^3$
Step 4: Mean Convex Subspaces $X_n$ in $\mathbb{H}^3$

- We want $T_{2n-1} \cup P_n$ to be uniquely minimizing in $X_n \subset \mathbb{H}^3$.

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- **Igloo Trick**: Let $\Pi_n = P_n^+ \# P_n^-$. Let $I_n$ be the component of $H^3 - \Pi_n$. 

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\( \Pi_1 \) is the least area plane in \( Y_1 = \mathbb{H}^3 - \mathcal{S}_1 \) where 
\( \partial_\infty \Pi_1 = \lambda_1 \). In particular, \( \Pi_1 = P_1 - \# \gamma_1 P_2 \) and \( \lambda_1 = \gamma_1 - \# \alpha_1 \gamma_2 \).

Similarly, one can iterate this process by using appropriate isometry \( \phi_n \) such that \( \lambda_n = \phi_n(\lambda_1) = \gamma_n - \# \alpha_n \gamma_{n+1}^+ \) is a simple closed curve in the region between \( \gamma_n \) and \( \gamma_{n+1} \). Here, \( \alpha_{2n-1}' \) is a line segment in the line \( x = -C \) with endpoints \((-C, r_{2n-1}, 0)\) and \((-C, r_{2n}, 0)\), while \( \alpha_{2n}' \) is a line segment in the line \( x = -C \) with endpoints \((-C, -r_{2n}, 0)\) and \((-C, -r_{2n+1}, 0)\). In particular, the bridges \( \alpha_{2n-1}' \) and \( \alpha_{2n}' \) are alternating sides (See Figure 6).

Then, let \( \Pi_n = \phi_n(\Pi_1) \) and \( \mathcal{R}_n = \phi_n(\mathcal{R}_1) \). Hence, define \( X_{n+1} = X_n - \mathcal{R}_n \). Notice that \( X_n \) is a mean convex subspace of \( \mathbb{H}^3 \).

Other than being mean convex, we will require one more property on \( X_2 \). By the construction of the least area plane \( \Pi_1 \sim P_1 - \# \alpha_1 P_2^+ \), for smaller choice of \( \rho \), we get a thinner bridge in \( \Pi_1 \) connecting \( P_1 \) and \( P_2 \). In particular, if \( \lambda_1^m = \gamma_1 - \# \alpha_1^m \gamma_2^+ \) is the simple closed curve obtained by connecting \( \gamma_1 \) and \( \gamma_2^+ \) along a bridge along \( \alpha_1' \) with thickness \( \rho_m \rightarrow 0 \), then let \( \Pi_1^m \) be the least area plane in \( Y_1 \) with \( \partial_\infty \Pi_1^m = \lambda_1^m \). By the construction, \( \Pi_1^m \rightarrow P_1 - P_2^+ \) as \( n \rightarrow \infty \).
Step 4: Mean Convex Subspaces $X_n$ in $\mathbb{H}^3$

- We want $T_{2n-1} \cup P_n$ to be uniquely minimizing in $X_n \subset \mathbb{H}^3$.

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- Let $X_1 = \mathbb{H}^3$ and $X_{n+1} = X_n - \mathcal{I}_n$. 

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- **Igloo Trick:** Let $\Pi_n = P_n^+ \# P_n^-$. Let $I_n$ be the component of $H^3 - \Pi_n$.
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**Lemma**

$T_{2n-1} \cup P_n$ is uniquely minimizing in $X_n$. 

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**Lemma**

$T_{2n-1} \cup P_n$ is uniquely minimizing in $X_n$.

**Theorem**

$T_{2n} = T_{2n-1} \# P_n$ is uniquely minimizing in $X_n$. 
Let $\Sigma = \lim T_n$. 
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Nonproper Minimal Surface of Desired Topology

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$\Sigma \sim \Sigma_2 \sim S$
Nonproper Minimal Surface of Desired Topology

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- $\Sigma \sim \Sigma_1 \#_\mu \Sigma_2$.
- $\Sigma \sim \Sigma_2 \sim S$.
- $\Sigma$ is nonproper as $\overline{\Sigma} \supset \overline{\Sigma_1} \supset P_\infty$. 

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Nonproper Minimal Surfaces with Arbitrary Topology in $H^3$
Final Remarks

- **The Bridge Principle at Infinity** for Complete Stable Minimal Surfaces and the Igloo Trick.

- Properly Embedded $H$-surfaces with arbitrary topology
  - Theorem \[C–\]
  - Any $S$ can be properly embedded in $H^3$ as a minimizing $H$-surface.

- Nonproperly Embedded $H$-surfaces with arbitrary topology
  - Unfortunately these techniques do not generalize to non-proper $H$-surfaces because of the orientation problem!
  - \[C–, Meeks, Tinaglia\] For $0 \leq H < 1$, $\exists$ a nonproperly embedded $H$-plane.
  - \[Meeks-Tinaglia\] For $H \geq 1$, Calabi-Yau Conjecture is true for $H$-surfaces in $H^3$. 
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