IUCSS

Benjamin R. Edwards
Indiana University

Finsler Geometry and Lorentz-Violating Field Theories

Overview

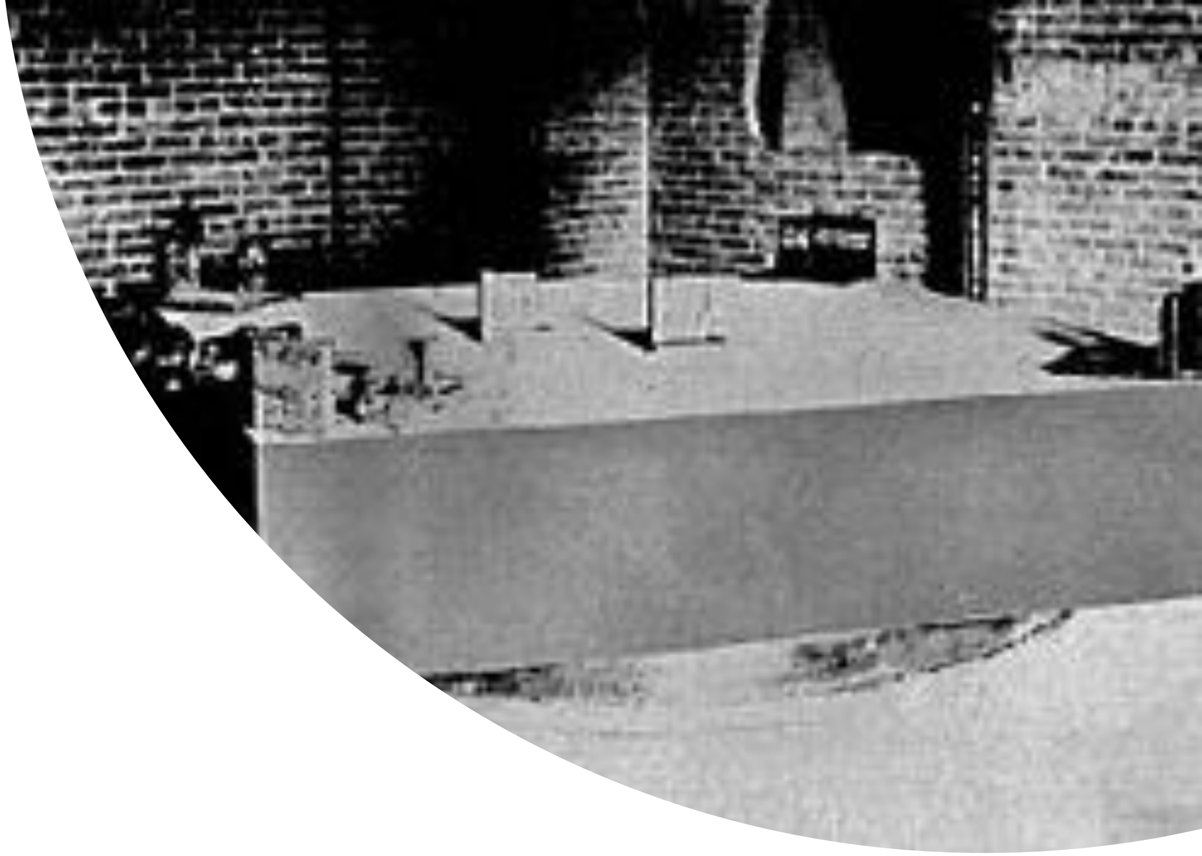
Introduction to Lorentz violation and the SME

- Scalar field theory
- Constructing point-particle lagrangians
- Associated Finsler spaces

Open questions

Introduction to Lorentz violation and the SME

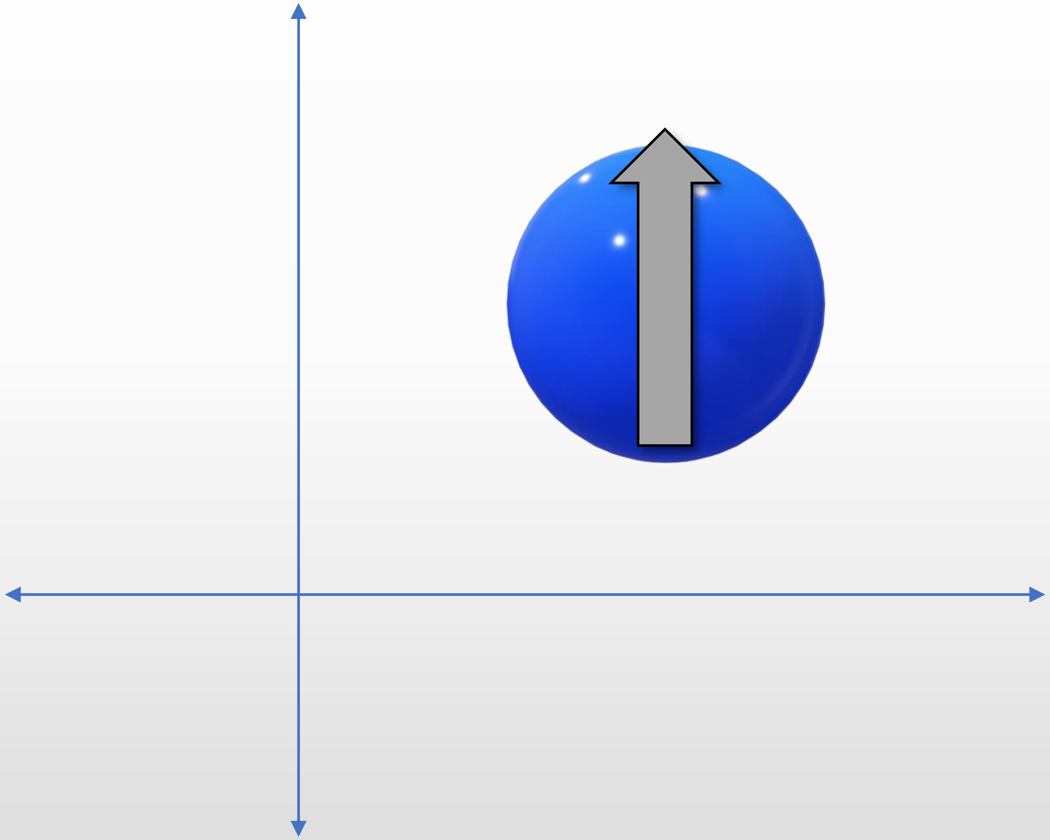
- What is Lorentz symmetry? What is Lorentz violation?
- How do we test Lorentz symmetry?
- Point-particle lagrangians from field theory
- Connection to Finsler geometry

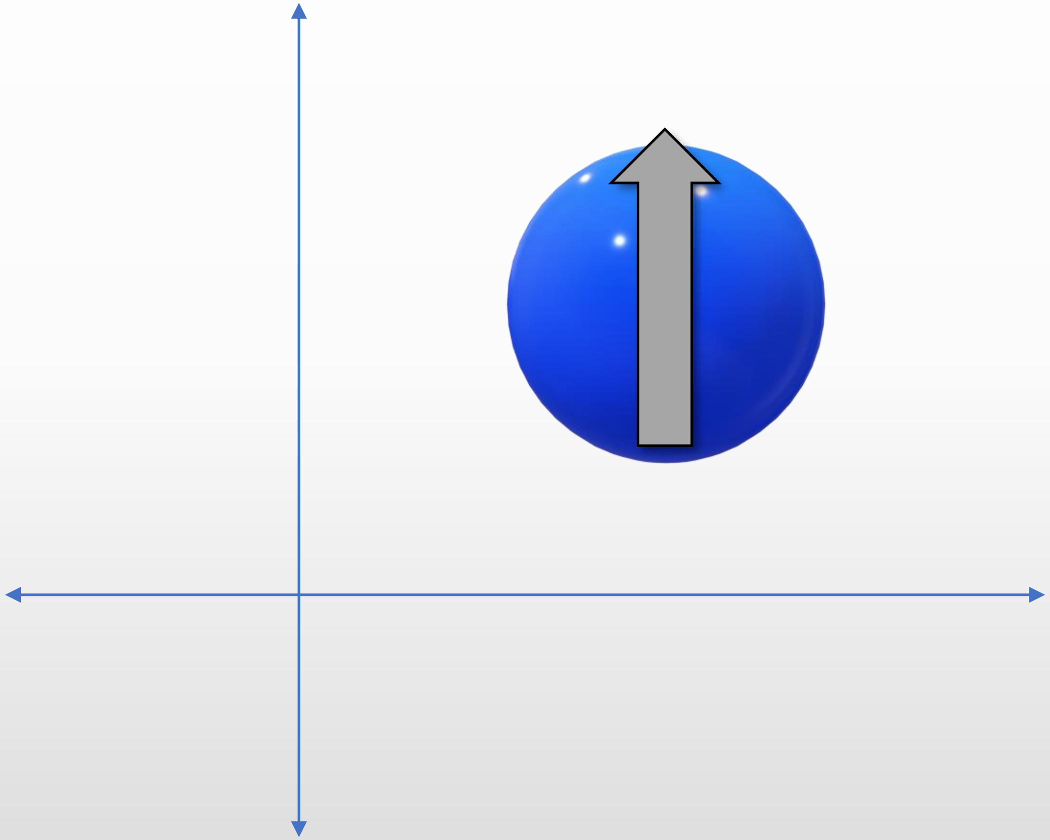


Michelson, Morley 1887

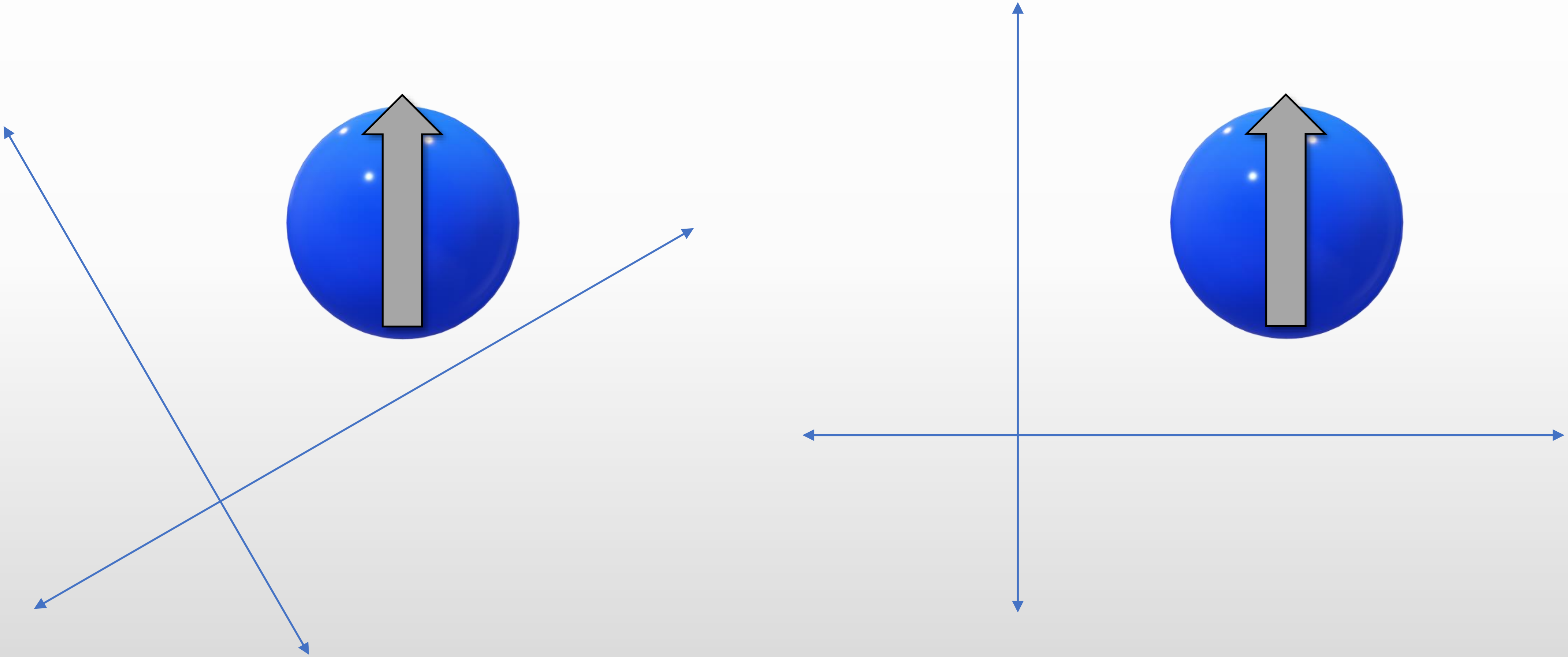
Lorentz symmetry

Lorentz symmetry



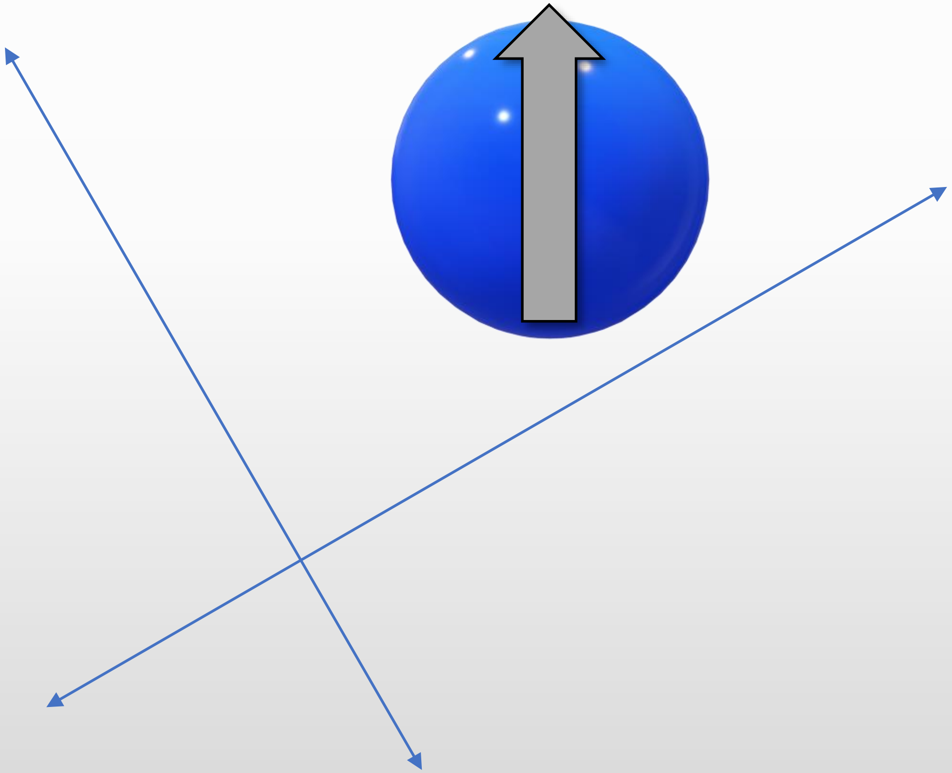


Lorentz symmetry

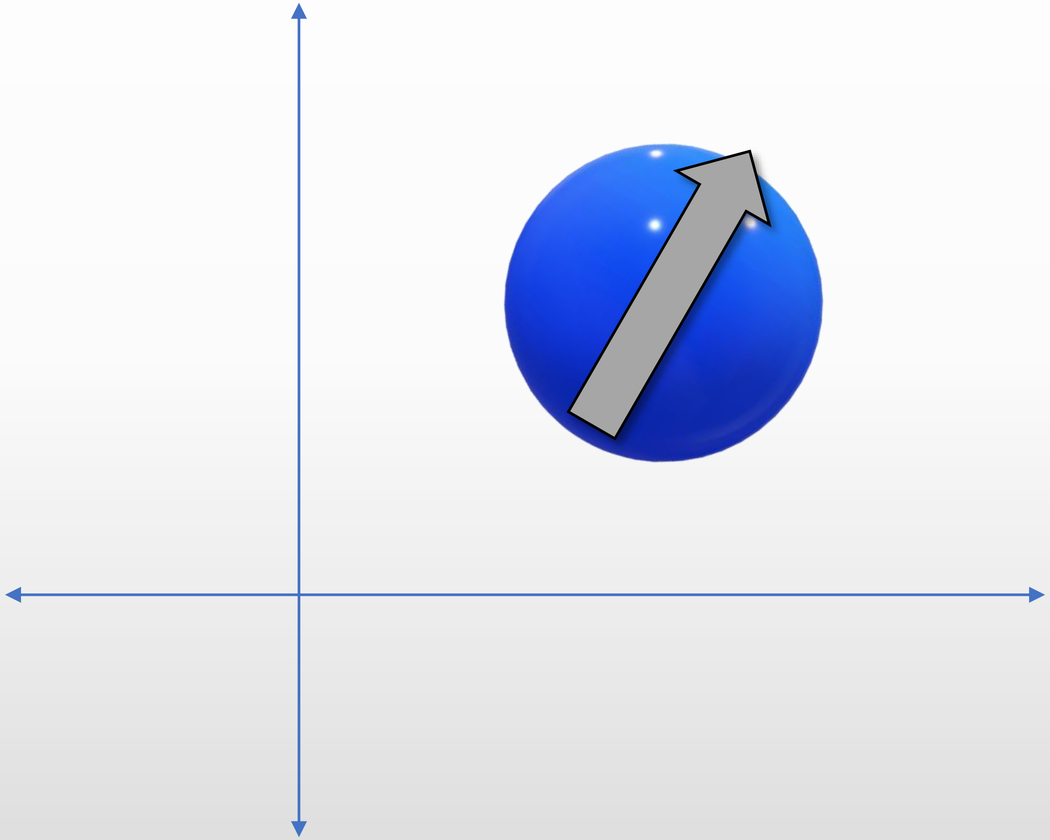


Coordinate transformation

Lorentz symmetry

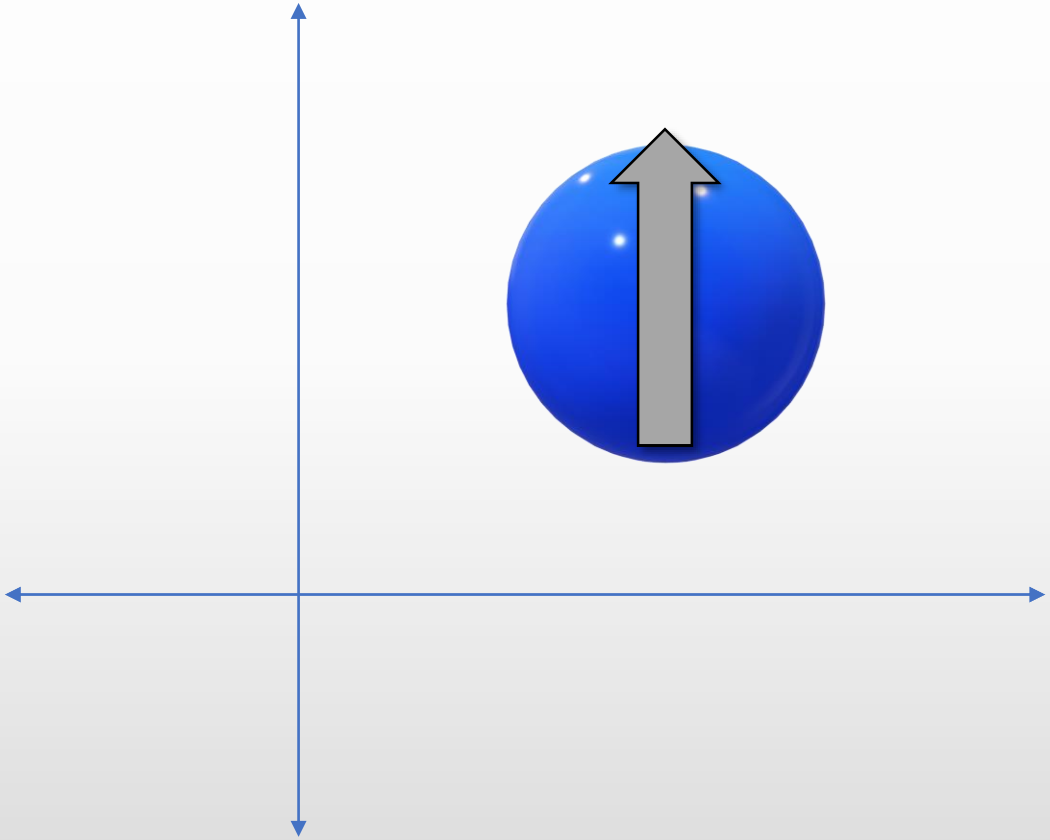


Coordinate transformation

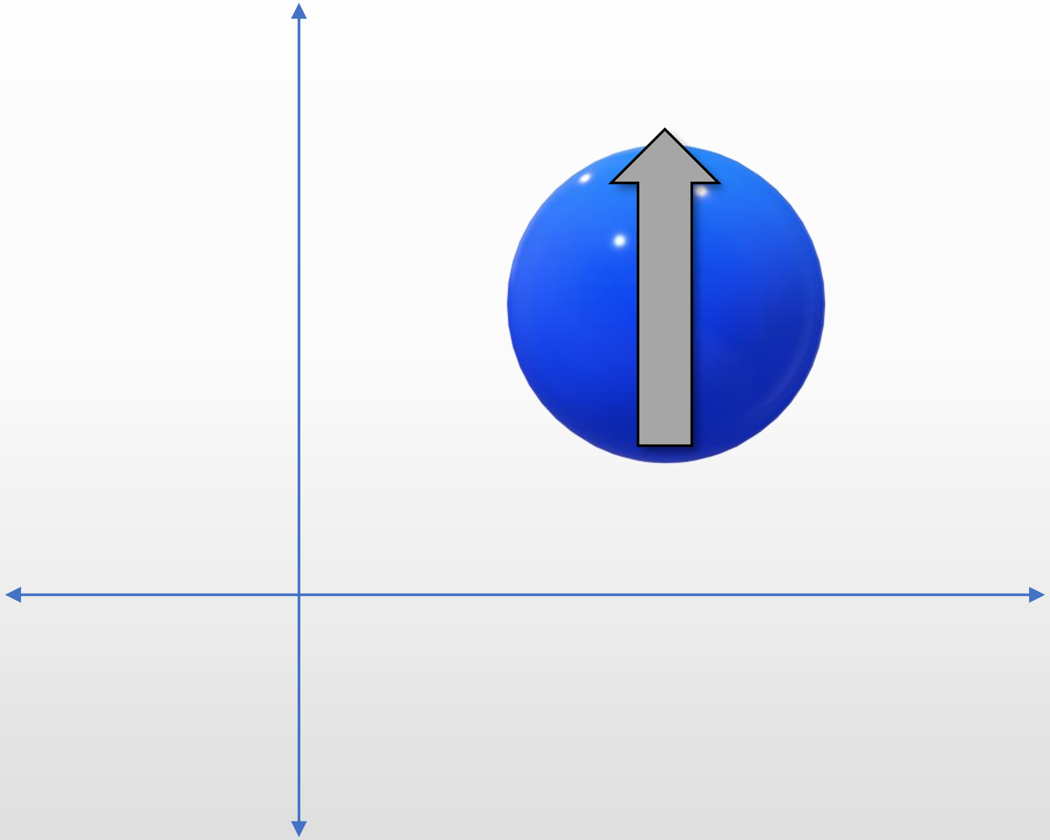


Particle transformation

Lorentz symmetry

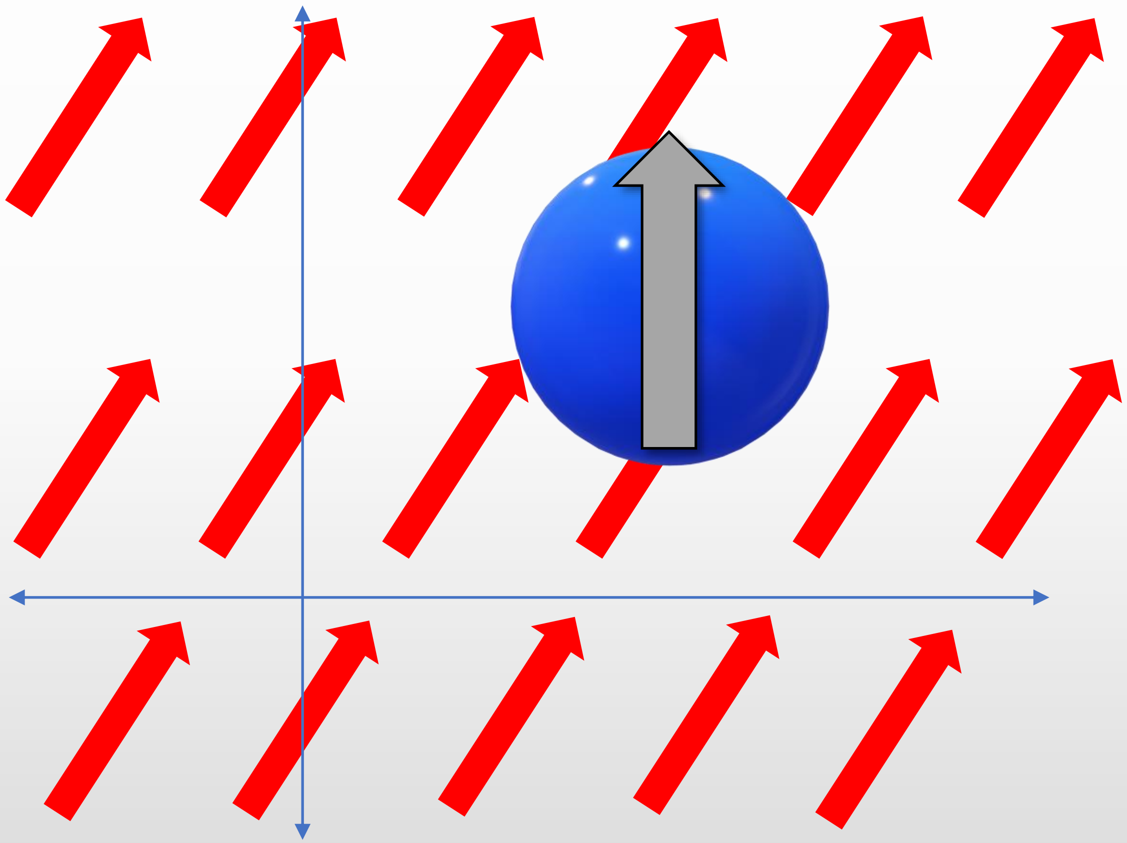


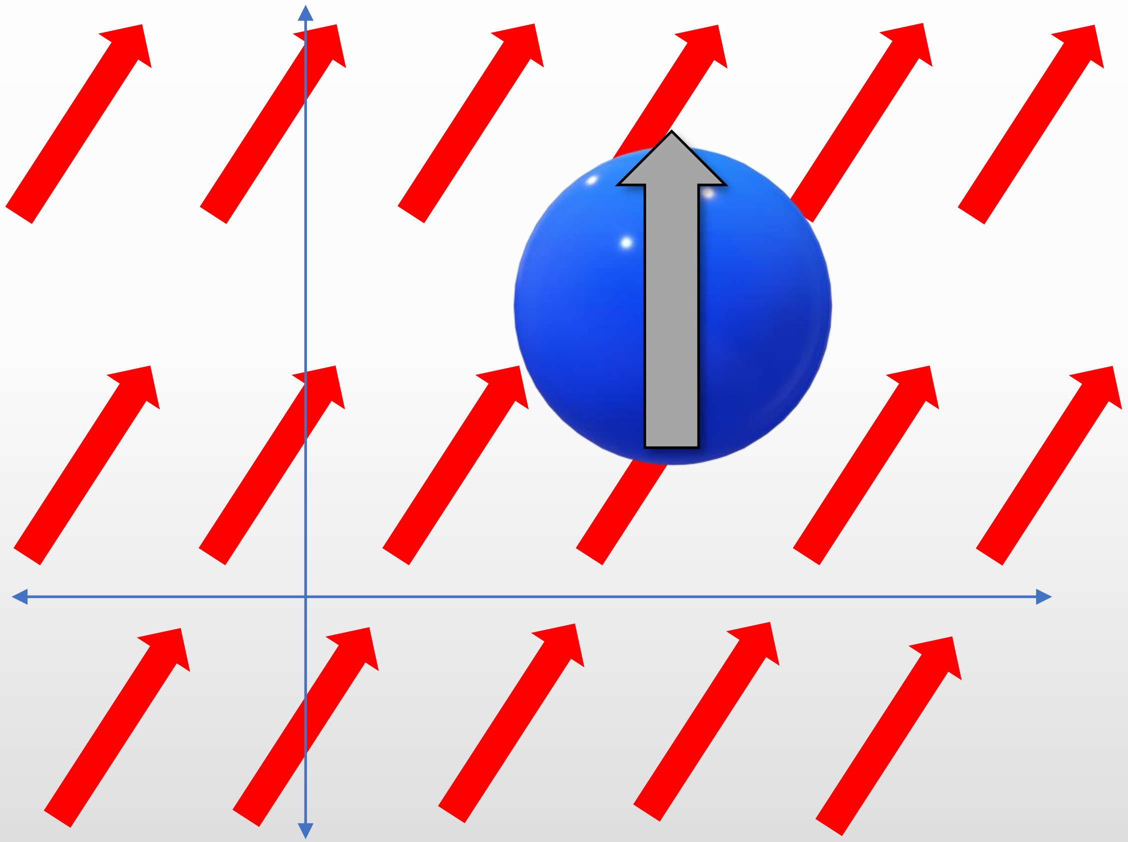
Coordinate transformation



Particle transformation

Lorentz violation

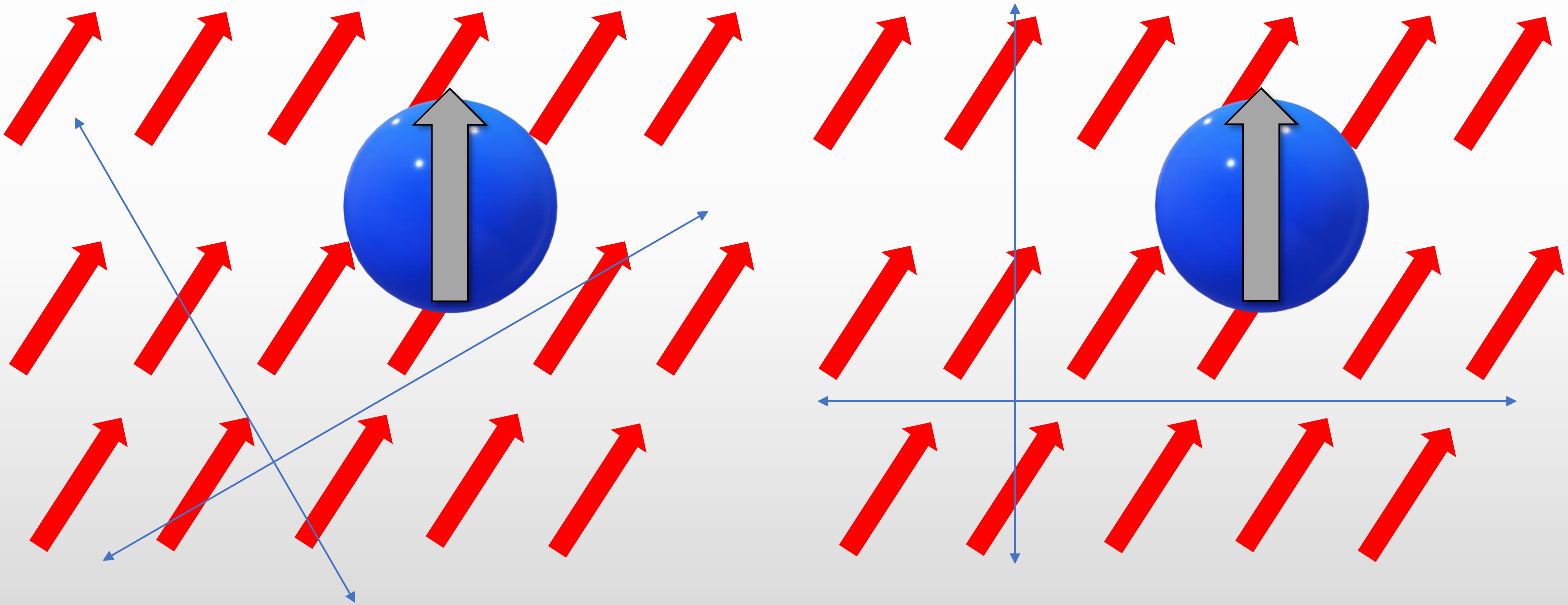




$U = -\mu \cdot B$

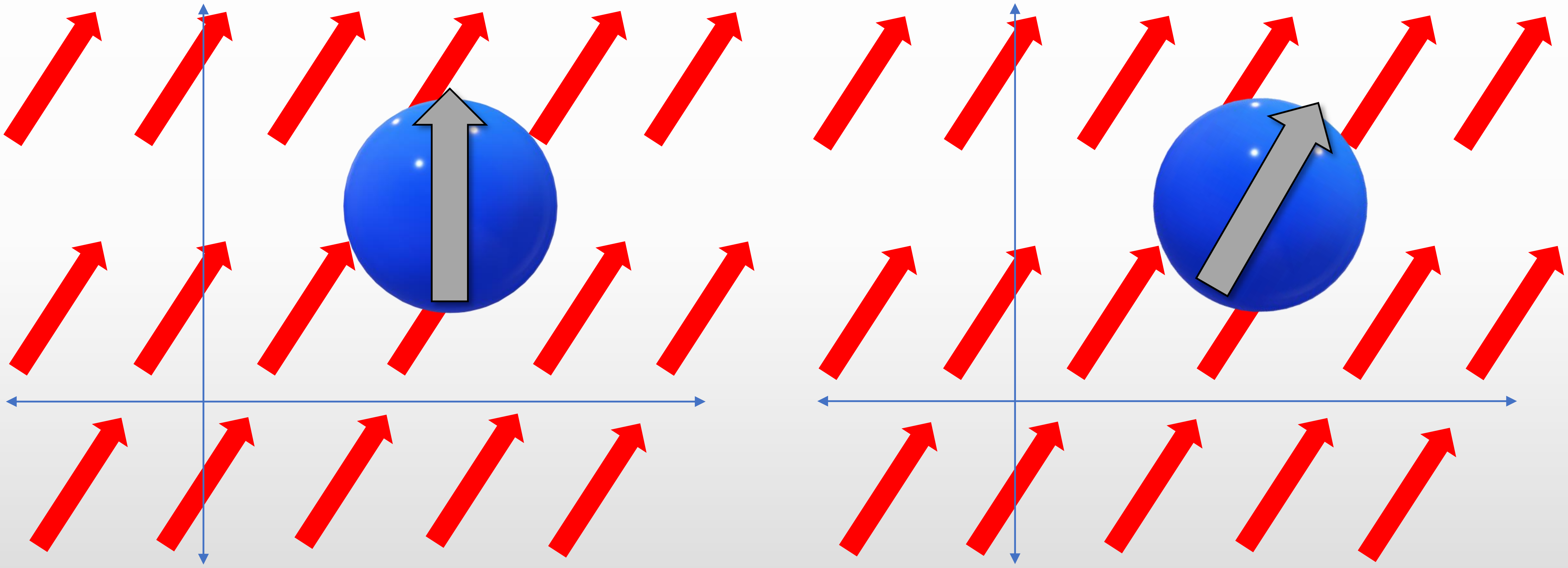
$U' = -\mu \cdot B$

Lorentz violation



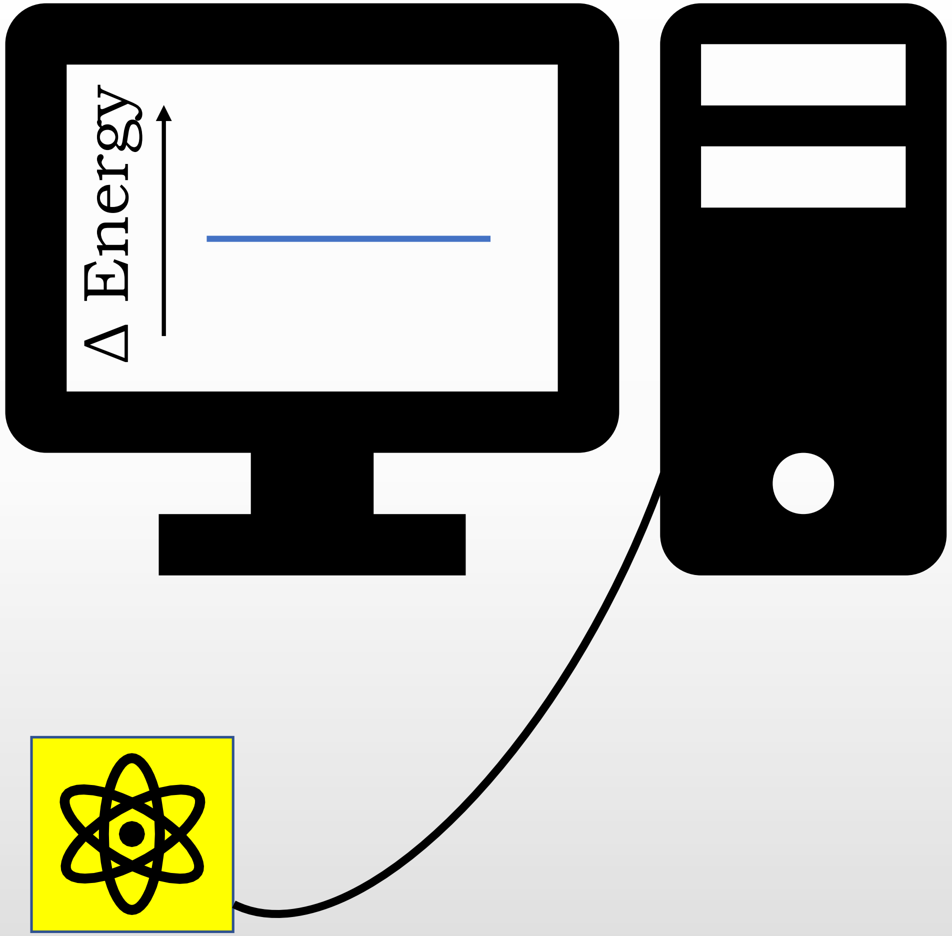
$U = -\mu \cdot B \quad = \quad U' = -\mu \cdot B$

Lorentz violation



$U = -\mu \cdot B$ $>$ $U' = -\mu \cdot B$

Meanwhile in the lab...



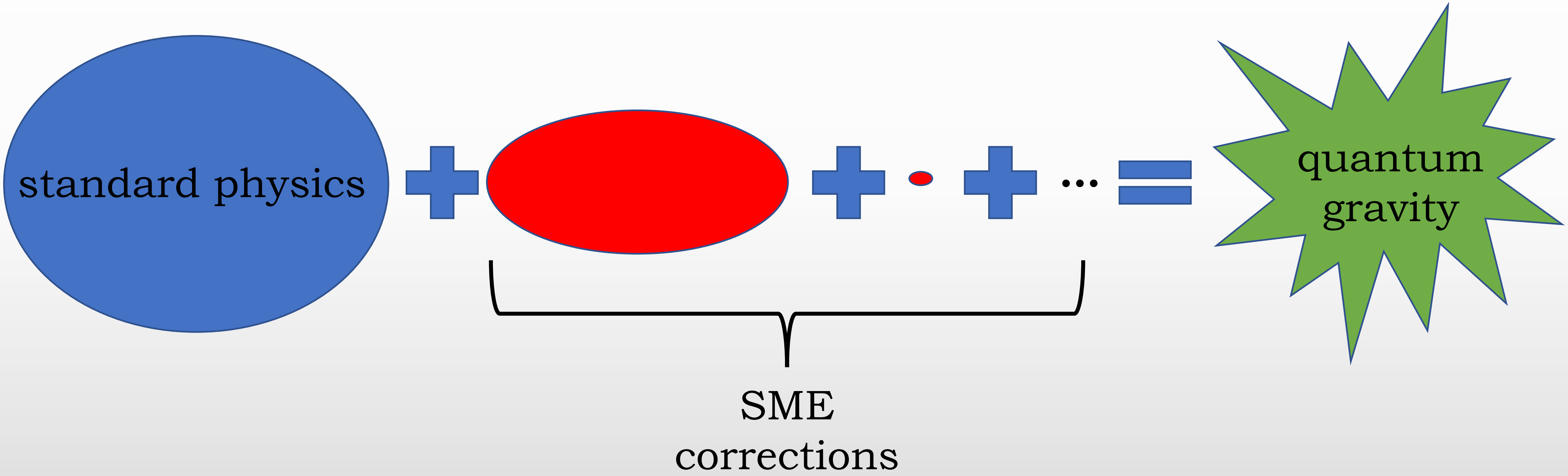
$$\mathcal{L}_{SME} \supset \frac{1}{2} i\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial_{\mu}}\psi - m\bar{\psi}\psi - a_{\mu}\bar{\psi}\gamma^{\mu}\psi - b_{\mu}\bar{\psi}\gamma_5\gamma^{\mu}\psi$$

- Coefficients for Lorentz violation act like background fields
- $\mathcal{L}_{SME}$ constructed from known fields
- Has implications for experiments at currently attainable energy levels

The Standard-Model Extension

Colladay, Kostelecký PRD 1997, 1998, Kostelecký PRD 2004

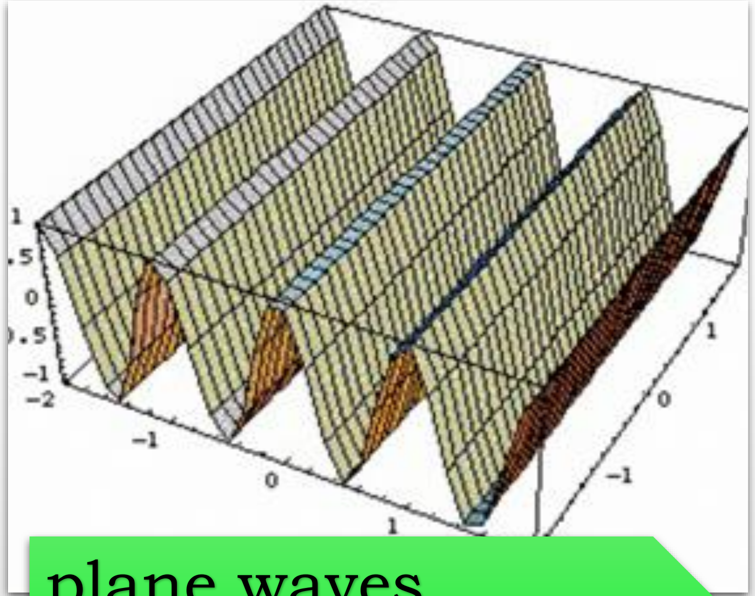
Effective field theory



From field theories to point particles

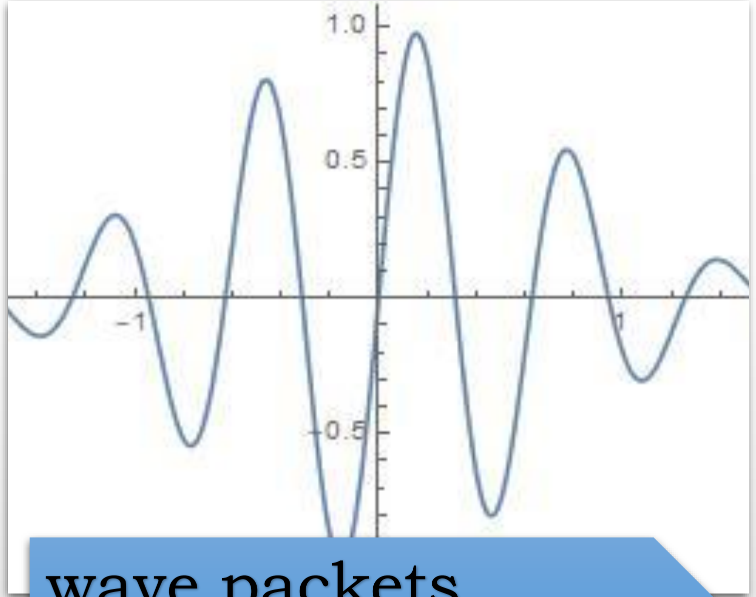
effective field theory

- successful explaining low-energy effects



plane waves

- building blocks of general solutions



wave packets

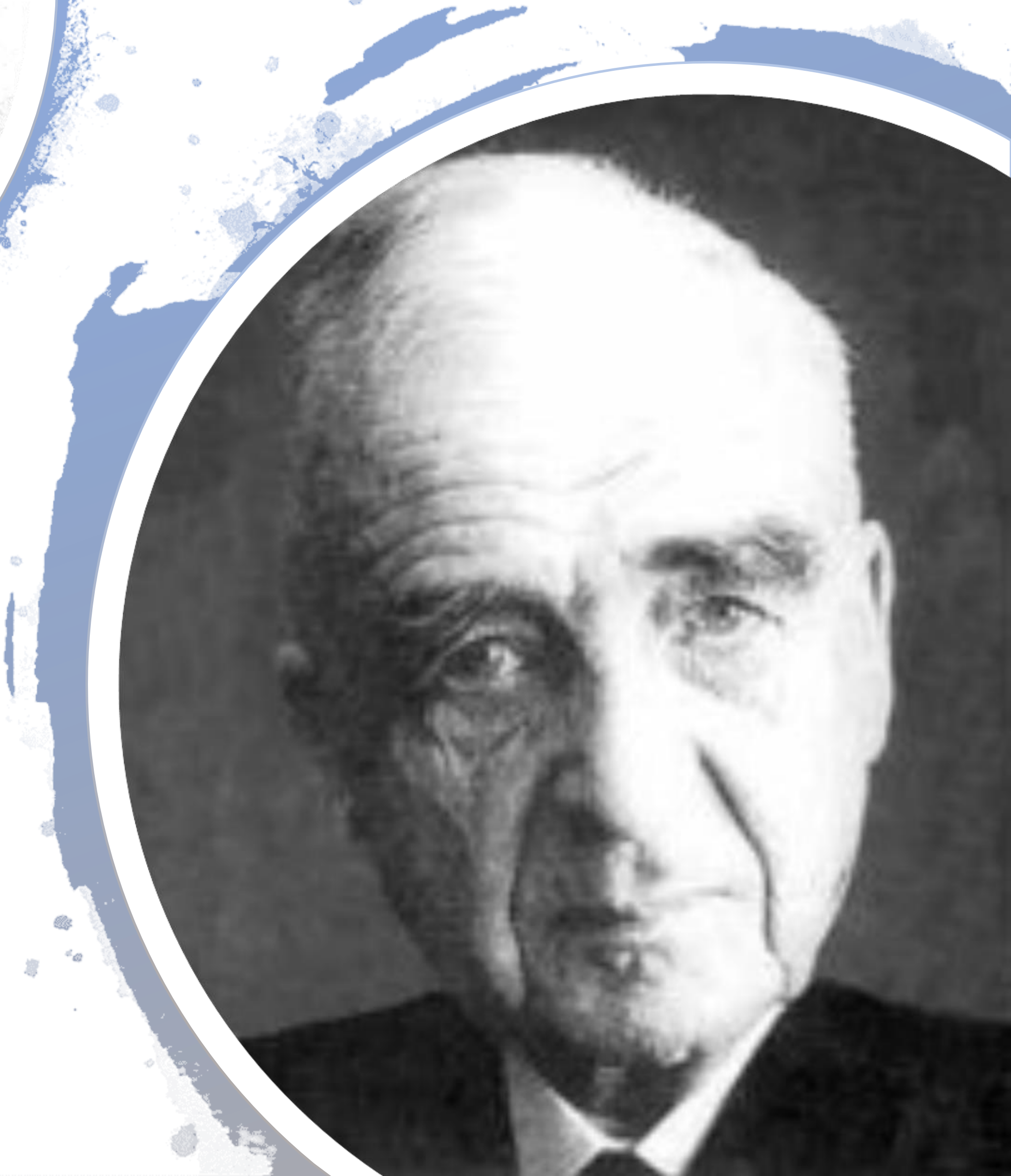
- follow classical trajectories

Why study point-particle lagrangians?

Little is known of kinematics in Lorentz-violating backgrounds

Many experiments involve signals from macroscopic bodies

Exact momentum-velocity relationship unknown



Connection to Finsler geometry

Lagrangians and Finsler norms

$L\colon \boldsymbol{TM} \to \mathbb{R}$

- $L(x,u)$ smooth on $TM\backslash S$
- L is 1-homogeneous in u
- Effective metric

$$g_{\mu\nu} = \frac{1}{2}\frac{\partial^2 L^2}{\partial u^\mu \partial u^\nu}$$

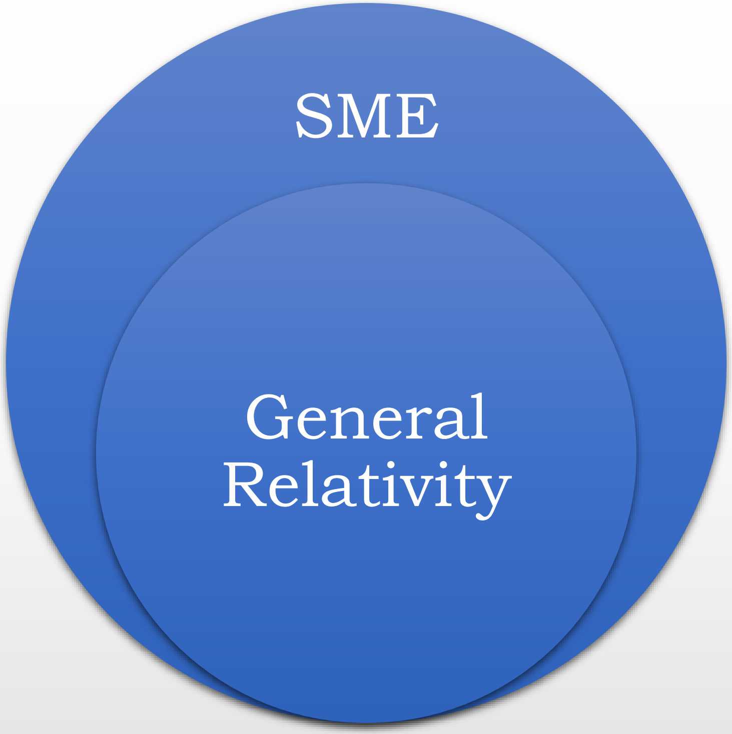
$\boldsymbol{F} : \boldsymbol{TM} \to [\mathbf{0}, \infty)$

- $F(x,y)$ smooth on $TM\backslash\{0\}$
- F is 1-homogeneous in y
- Finsler metric (positive definite!)

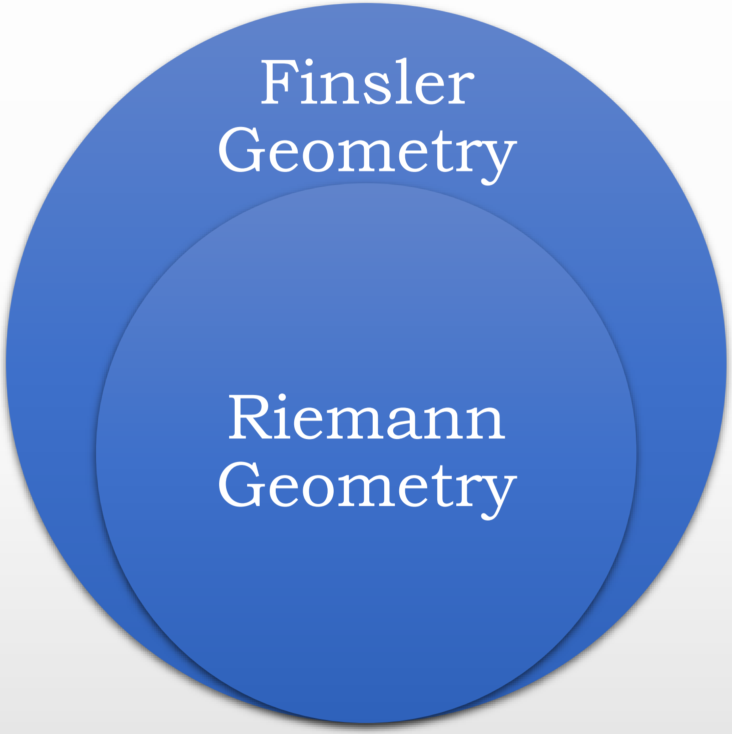
$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

Goldstien Classical Mechanics 1950, Bao, Chern, Shen An Introduction to Riemann-Finsler Geometry 2000

Finsler geometry

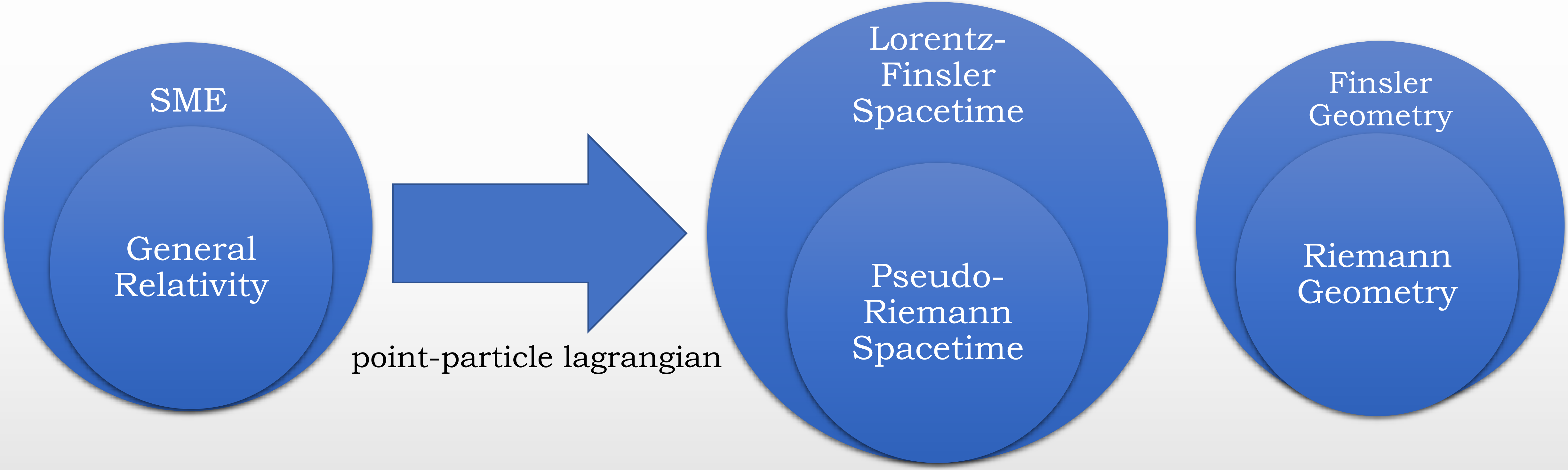


Constructed by adding background field couplings

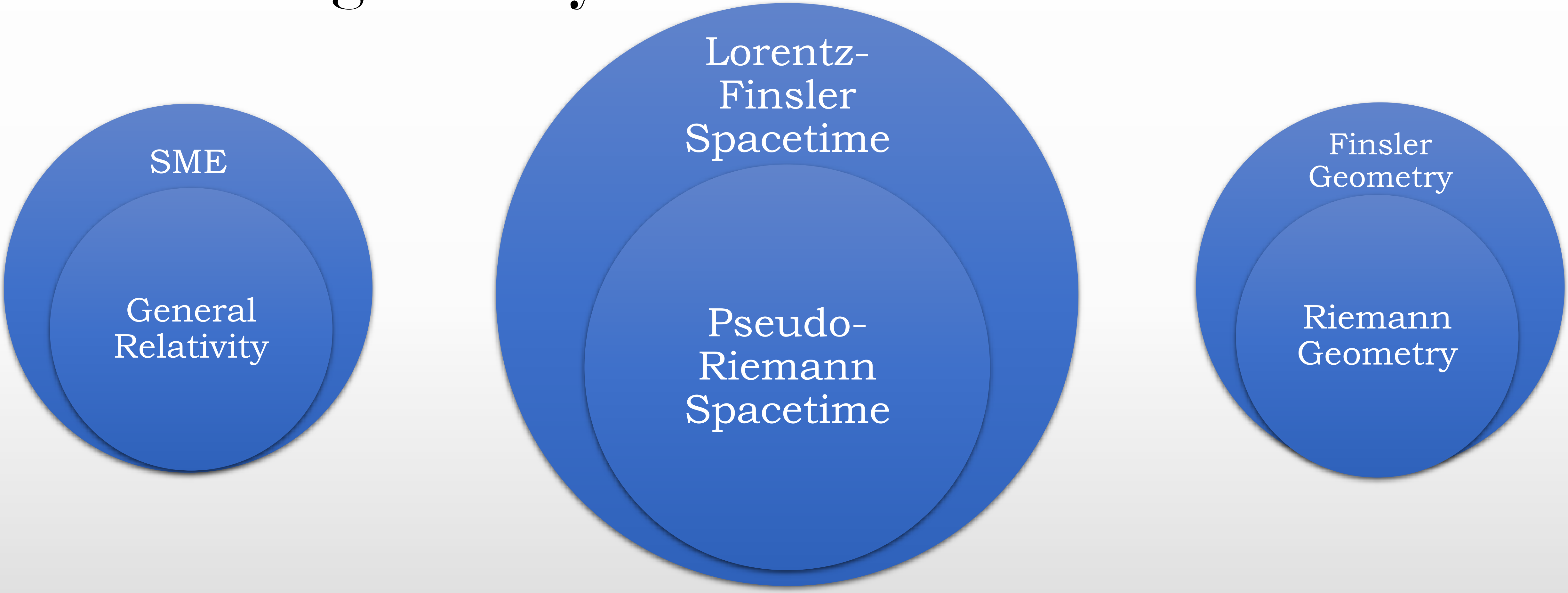


Constructed by adding covector field couplings

Finsler geometry



Finsler geometry

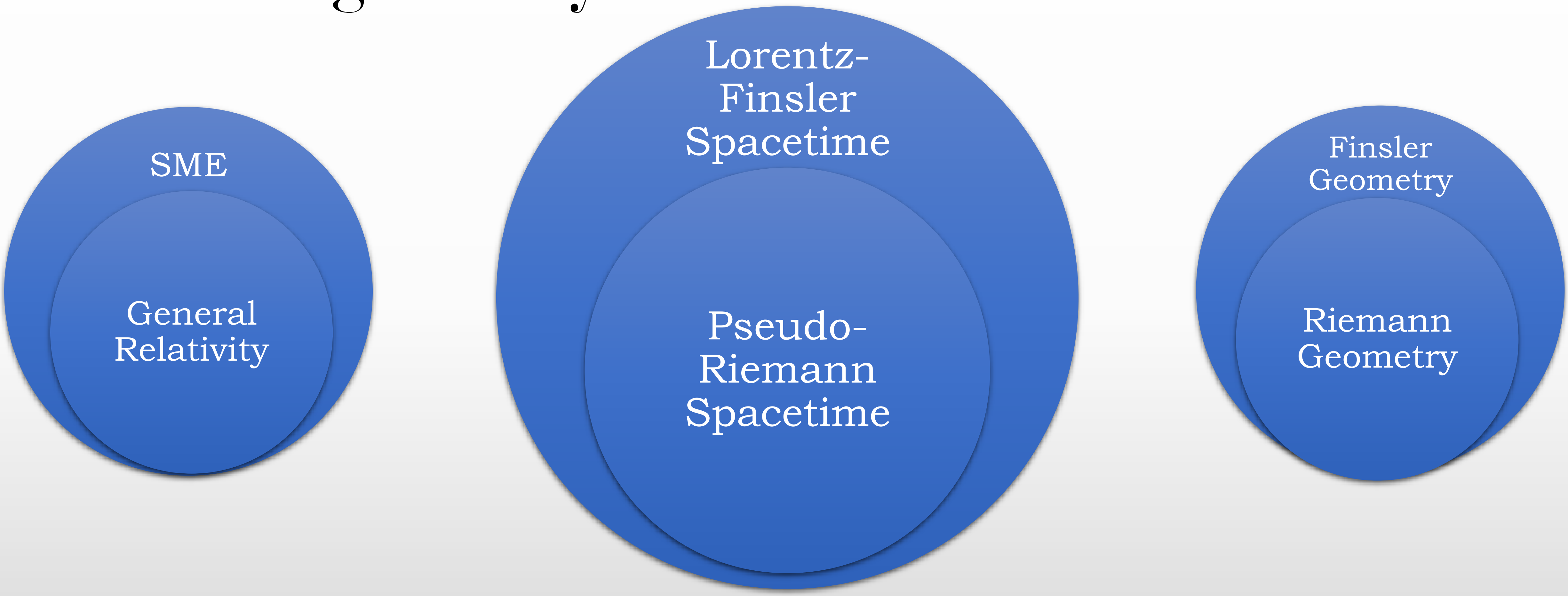


precise definition still open

Beem Canad. J. Math. 1970, Asanov Finsler Geometry, Relativity, and Gauge Theories 1985

Miron, Anastasiei The theory of Lagrange Spaces: Theory and Applications 1994

Finsler geometry



precise definition still open

Benjacu, Farran Geometry of Pseudo-Finsler Submanifolds 2000, Pfeifer, Wohlfarth PRD 2011, Kostelecký PLB 2011, Lämmerzahl, Perlick, Hasse PRD 2012, Javaloyes, Sánchez arXiv: 1805.06978

Why study point-particle lagrangians?

Little is known of kinematics in Lorentz-violating backgrounds

Many experiments involve signals from macroscopic bodies

Exact momentum-velocity relationship unknown

Why study point-particle lagrangians?

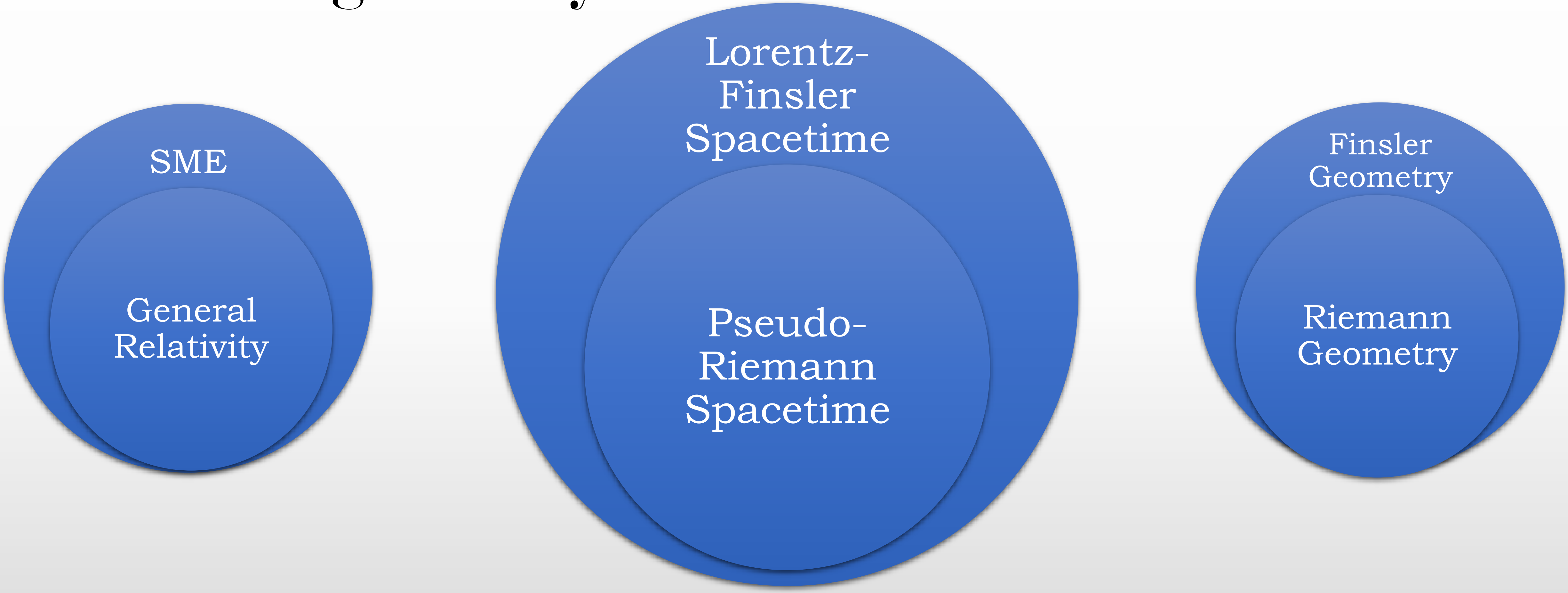
Provide us with more physical examples of possible Lorentz-Finsler spacetimes

Little is known of kinematics in Lorentz-violating backgrounds

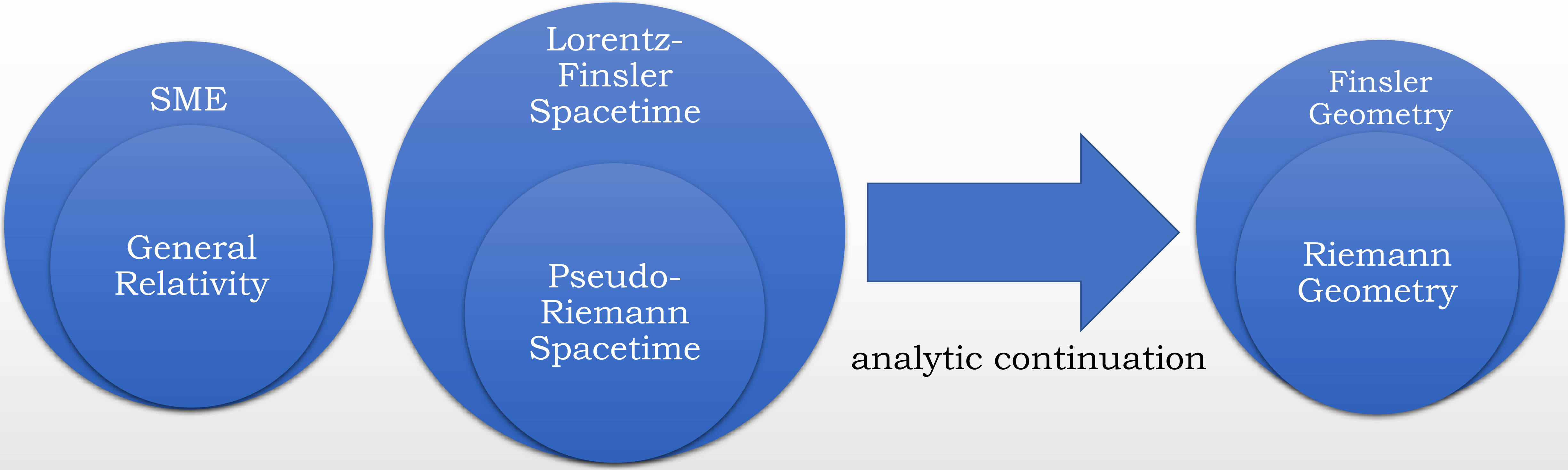
Many experiments involve signals from macroscopic bodies

Exact momentum-velocity relationship unknown

Finsler geometry



Finsler geometry



Spontaneous vs. explicit symmetry breaking

- Spontaneous
  - underlying theory has Lorentz symmetry
  - dynamic fields acquire backgrounds
- Explicit
  - nondynamic background
  - in GR this has implications for geometry

Kostelecký, Samuel PRD 1989, Kostelecký PRD 2004

Explicit breaking and gravity

- Einstein tensor and Bianchi identity

$$D_\mu G^{\mu\nu} = 0$$

- Einstein equation

$$G^{\mu\nu} = \kappa T^{\mu\nu}$$

- Together these imply

$$D_\mu T^{\mu\nu} = 0$$

Explicit breaking and gravity

- Equations of motion in the presence of background fields imply

$$D_\mu T^{\mu\nu} = J_x D^\nu k^x$$

- To be consistent, right hand side must vanish!

- Need extra degrees of freedom to reconcile these requirements
  - these may come from the direction dependence in Finsler geometry!

Kostelecký PRD 2004

Finsler geometry

Geodesics are controlled by a metric with extra degrees of freedom

$$g_{jk} = r_{jk} + \ldots$$

Riemann metric
(depends on point in manifold)

modification
(can also depend on direction!)

Conjecture: SME background fields may give rise to these modifications

Finsler (thesis) 1918 , Kostelecký PRD 2004

Overview

- Introduction to Lorentz violation and the SME
- Scalar field theory
- Constructing point-particle lagrangians
- Associated Finsler spaces
- Open questions

Scalar field theory

Why scalars?

The most general Lorentz-violating scalar field theory has not been studied

All particles in nature exhibit a property called spin

- scalar, spinor, vector,...

A quantum scalar field theory describes spin 0 particles

Why scalars?

In certain cases, spin can complicate the trajectory

A large subset of Lorentz-violating effects are spin independent

Free particles can be handled as if they have 0 spin in this case

General scalar field theory

$$\mathcal{L}(\phi, \phi^\dagger) = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi + \underbrace{\partial_\mu \phi^\dagger (\widehat{k}_c)^{\mu\nu} \partial_\nu \phi}_{} - \tfrac{1}{2} \underbrace{[i \phi^\dagger (\widehat{k}_a)^\mu \partial_\mu \phi + \text{h.c.}]}_{}$$

even number of derivatives

$$(\widehat{k}_c)^{\mu\nu} = \sum_{d=n} (i)^{d-n} (k_c^{(d)})^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{d-n}} \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_{d-n}}$$

odd number of derivatives

$$(\widehat{k}_a)^{\mu} = \sum_{d=n-1} (i)^{d-n+1} (k_a^{(d)})^{\mu\alpha_1\alpha_2\cdots\alpha_{d-n+1}} \partial_{\alpha_1} \partial_{\alpha_2} \cdots \partial_{\alpha_{d-n+1}}$$

k's have constant cartesian components

n: number of spacetime dimensions

d: mass dimension of the operator

Edwards, Kostelecký PLB 2018

Existing work in scalar field theory

Much work done in minimal sector

Borges, Ferrari, Farone arXiv:1809.08883
Xiao PRD 2018
de Paula Netto PRD 2018
Silva, Carvalho IJGMMP 2018
Scarpelli, Brito, Felipe, Nascimento, Petrov EPJC 2017
Cruz, Bezerra de Mello, Petrov PRD 2017, MPLA 2018
Kamand, Altschul, Schindler PRD 2017
Casana, da Silva MPLA 2015
Carvalho PLB 2013, PLB 2014
Altschul PRD 2013
Ferrero, Altschul PRD 2011
Bazeia, Barreto, Menezes PRD 2006
Altschul, PLB 2006
Anderson, Sher, Turan PRD 2004
Berger, Kostelecký PRD 2002
Colladay, Kostelecký PRD 1997, 1998

Existing work in scalar field theory

- Previous work all $n = 4, d = 3\ ,\ 4$
- Recent work on $n = 4\ , d = 6$
- No results for $n \neq 4$
- Nonminimal sector largely ignored

Nascimento, Petrov, Reyes EPJC 2018

General scalar field theory

$$\mathcal{L}(\phi, \phi^\dagger) = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi + \partial_\mu \phi^\dagger (\widehat{k}_c)^{\mu\nu} \partial_\nu \phi - \tfrac{1}{2}[i\phi^\dagger (\widehat{k}_a)^\mu \partial_\mu \phi + \text{h.c.}]$$

- $d = n$ absorbed into metric, $d = n - 1$ is a local phase
  - only $d > n$ can be observed for single scalar field!
- Coefficients can be taken to be traceless, symmetric
- In 4 dimensions, k_c is CPT even, k_a is CPT odd
  - hermitian fields cannot violate CPT!

Edwards, Kostelecký PLB 2018

Equations of motion

$$(\partial^\mu \partial_\mu + m^2 + i(\widehat{k}_a)^\mu \partial_\mu + (\widehat{k}_c)^{\mu\nu} \partial_\mu \partial_\nu)\phi = 0$$

lead to a dispersion relation

$$p^2 - m^2 - (\widehat{k}_a)^\mu p_\mu + (\widehat{k}_c)^{\mu\nu} p_\mu p_\nu = 0$$

where extra terms affect propagation

Edwards, Kostelecký PLB 2018

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Constructing point-particle lagrangians

Equations of motion

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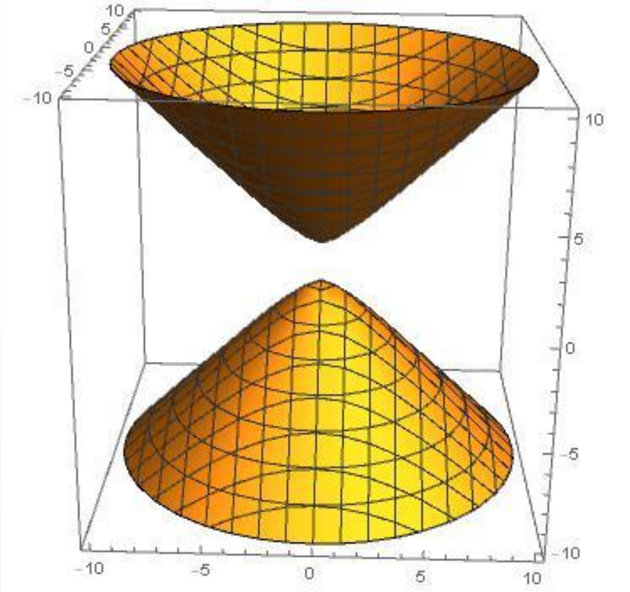
Edwards, Kostelecký PLB 2018

Two ways to view dispersion relations

$$p^2 - m^2 - (\widehat{k}_a)^\mu p_\mu + (\widehat{k}_c)^{\mu\nu} p_\mu p_\nu = 0$$

Field theory

Wave vectors constrained to a hypersurface



Point particles

Momentum constrained to a hypersurface

Can we build a point-particle lagrangian that generates this dispersion?

Constructing point-particle lagrangians

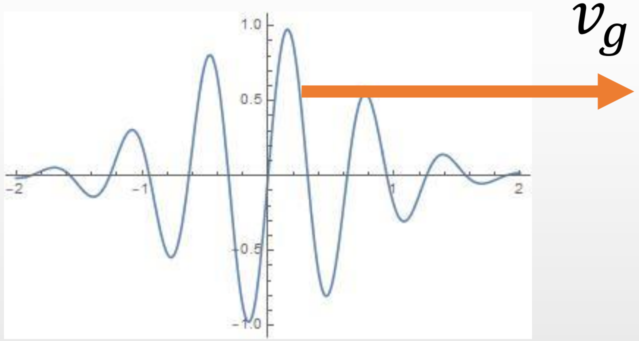
- Start with a dispersion relation

$$R(p) = 0$$

- Enforce wave-packet group velocity = classical velocity

$$v_g = \frac{d\omega}{dp}$$

$$\frac{\partial p_0}{\partial p_j} = -\frac{u^j}{u^0} \quad ; \quad j = 1,2,\ldots,n-1$$



- Action invariant under reparameterization: $L(\lambda u) = \lambda L(u)$

$$\Rightarrow L = -u^\mu p_\mu$$

Kostelecký, Russell PLB 2010

Constructing point-particle lagrangians

- n equations can eliminate n momentum components
- In principle, $L = -u^\mu p_\mu$ becomes $L = L(u)$
- In practice, only known exactly for:
  - quadratic dispersions
  - quartic dispersions
- Calculations are difficult in general

Kostelecký, Russell PLB 2010

Constructing point-particle lagrangians

- Formalism applied to
  - *face*, *ab*, *H* limits of SME
  - exact lagrangians found for minimal coefficients

- Ansatz method used in fermion sector of SME
  - results are for nonminimal coefficients to first order

- New method generates all orders for minimal and nonminimal terms

Kostelecký, Russell PLB 2010, Reis, Schreck PRD 2018, Edwards, Kostelecký PLB 2018

Extended method applied to scalars

Can find exact lagrangian for $d = n$

$$p_\mu \Omega^{\mu\nu} p_\nu = m^2 \qquad \Omega = \eta + k^{(n)}$$

$$(u^0 \Omega^{j\nu} - u^j \Omega^{0\nu}) p_\nu = 0$$

$$u^\mu = -\frac{L^{(n)}}{m^2} \Omega^{\mu\nu} p_\nu$$

Edwards, Kostelecký PLB 2018

Extended method applied to scalars

Can find exact lagrangian for $d = n$

$$p_\mu \Omega^{\mu\nu} p_\nu = m^2$$

$$(u^0 \Omega^{j\nu} - u^j \Omega^{0\nu}) p_\nu = 0$$

$$u^\mu = -\frac{L^{(n)}}{m^2} \Omega^{\mu\nu} p_\nu$$

matches previous results

$$L^{(n)} = -m\sqrt{u^\mu (\Omega^{-1})_{\mu\nu} u^\nu}$$

Kostelecký, Russell PLB 2010, Edwards, Kostelecký PLB 2018

Extended method applied to scalars

- For $d > n$, extra p dependence makes obtaining exact solution difficult
- Following previous steps produces a power series for L after expansion

$$L^{(d)}(u,p,k^{(d)}) = \tfrac{1}{4}(d-n+2)u_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{d-n+2}}(k^{(d)})^{\alpha_1\cdots\alpha_{d-n+2}}$$
$$+ m\bar{u}\sum_{s=0}^{j}\sum_{j}[(-1)^j a_{js}((d-n+2)u_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{d-n+2}}(k^{(d)})^{\alpha_1\cdots\alpha_{d-n+2}})^{2s}$$
$$((d-n)p_{\alpha_1}\cdots p_{\alpha_{d-n+2}}(k^{(d)})^{\alpha_1\cdots\alpha_{d-n+2}})^{j-s}]$$

$$a_{js} = \frac{(2j)!}{m^{2j}u^{2s}(2j-1)8^{j+s}j!s!(j-s)!}$$

Edwards, Kostelecký PLB 2018

Extended method applied to scalars

$$L^{(d)}(u,p,k^{(d)}) = \tfrac{1}{4}(d-n+2)u_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{d-n+2}}(k^{(d)})^{\alpha_1\cdots\alpha_{d-n+2}}$$

$$+\, m\bar{u}\sum_{s=0}^{j}\sum_{j}[(-1)^j a_{js}((d-n+2)u_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{d-n+2}}(k^{(d)})^{\alpha_1\cdots\alpha_{d-n+2}})^{2s}$$

$$((d-n)p_{\alpha_1}\cdots p_{\alpha_{d-n+2}}(k^{(d)})^{\alpha_1\cdots\alpha_{d-n+2}})^{j-s}]$$

Define zeroth order

$$L_0^{(d)} \equiv L^{(d)}(u,p,0) = -m\bar{u} \qquad \bar{u} = \sqrt{u^\mu \eta_{\mu\nu} u^\nu}$$

Edwards, Kostelecký PLB 2018

Extended method applied to scalars

$$L_0^{(d)} \equiv L^{(d)}(u, p, 0) = -m\bar{u}$$

Canonical momentum given by

$$(p_0^{(d)})_\mu = -\frac{\partial L_0^{(d)}}{\partial u^\mu} = m\hat{u}_\mu \qquad \hat{u}^\mu = \frac{u^\mu}{\bar{u}}$$

Edwards, Kostelecký PLB 2018

Extended method applied to scalars

$$(p_0^{(d)})_\mu = -\frac{\partial L_0^{(d)}}{\partial u^\mu} = m\hat{u}_\mu$$

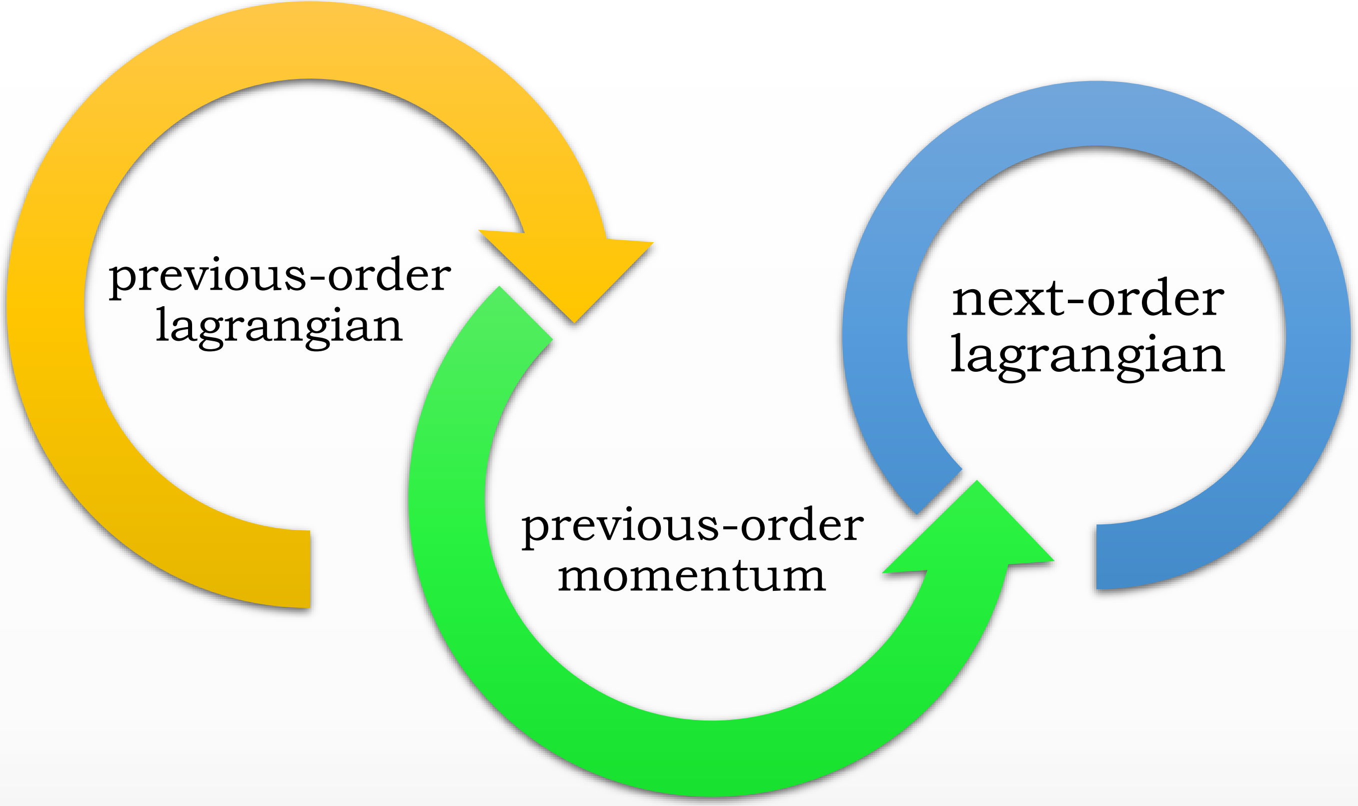
$$L^{(d)}(u, p, k^{(d)})$$

Reinsert to get next-order L

$$L_1^{(d)} = L^{(d)}(u, p_0, k^{(d)}) = -m\bar{u}\left[1 - \tfrac{1}{2}m^{d-n}(\hat{u}^{\alpha_1} \cdots \hat{u}^{\alpha_{d-n+2}}(k^{(d)})_{\alpha_1 \cdots \alpha_{d-n+2}})\right]$$

Edwards, Kostelecký PLB 2018

Iterative process



$$L_3^{(d)} = L_0^{(d)}\big[1 - \tfrac{1}{2}\widetilde{k}^{(d)} - \tfrac{1}{8}(d-n+1)^2(\widetilde{k}^{(d)})^2 + \tfrac{1}{8}(d-n+2)^2\ \widetilde{k}^{(d)}{}_{\alpha}\ \widetilde{k}^{(d)\alpha} - \tfrac{1}{16}(d-n+1)^4(\widetilde{k}^{(d)})^3$$
$$+\tfrac{1}{16}(d-n+1)(2d-2n+1)\ \widetilde{k}^{(d)}\ \widetilde{k}^{(d)}{}_{\alpha}\ \widetilde{k}^{(d)\alpha} - \tfrac{1}{16}(d-n+1)(d-n+2)^2\ \widetilde{k}^{(d)}{}_{\alpha}\ \widetilde{k}^{(d)\alpha\beta}\ \widetilde{k}^{(d)}{}_{\beta}\big]$$

$$\widetilde{k}^{(d)}{}_{\alpha_1\cdots\alpha_s} = (k^{(d)})_{\alpha_1\cdots\alpha_s\alpha_{s+1}\cdots\alpha_{d-n+2}}\hat{u}^{\alpha_{s+1}}\cdots\hat{u}^{\alpha_{d-n+2}}$$

Matches previous results derived from fermion limit of SME

Reis, Schreck PRD 2018, Edwards, Kostelecký PLB 2018

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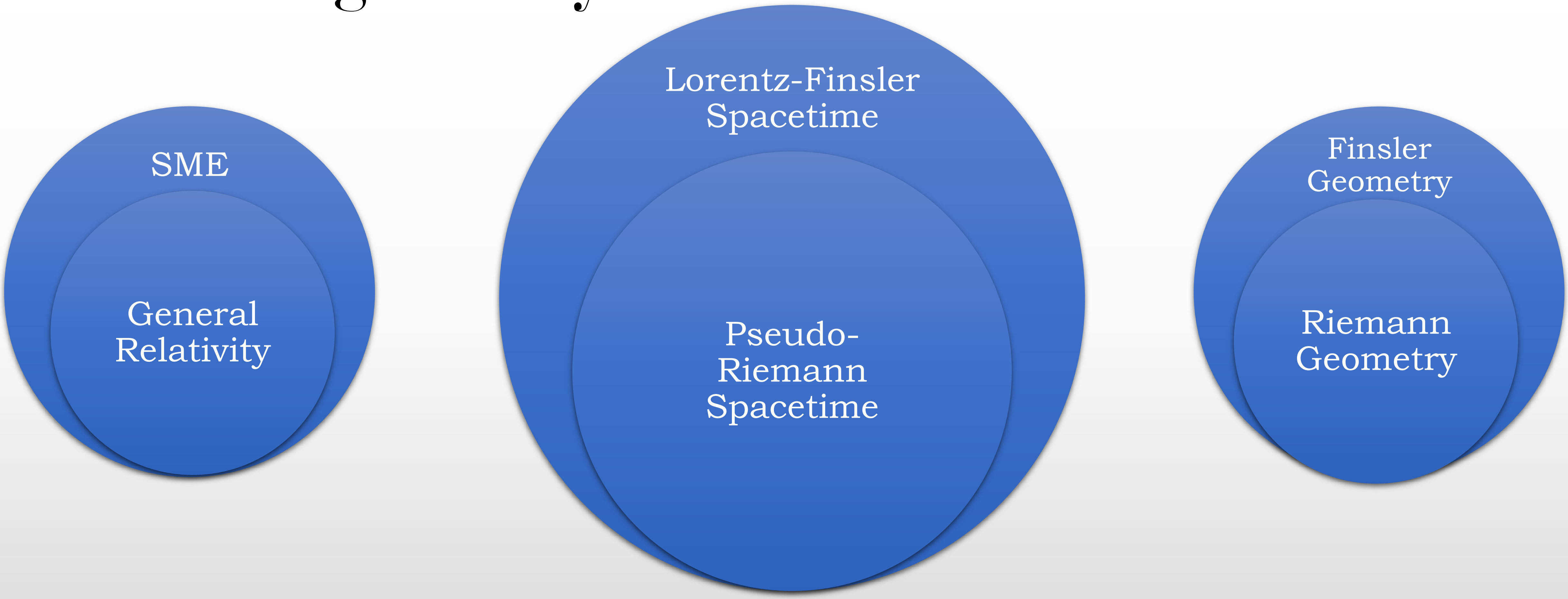
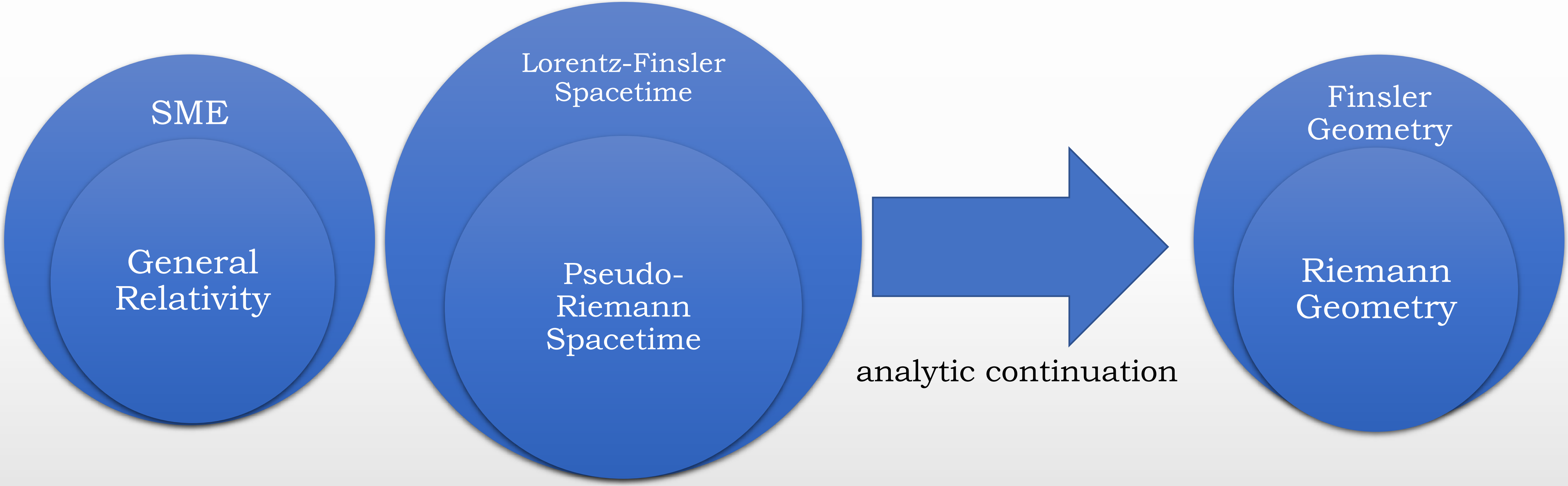
Associated Finsler spaces

Finsler geometry



Finsler geometry



Lagrangians and Finsler norms

$L\colon TM \to \mathbb{R}$

- $L(x, u)$ smooth on $TM \backslash S$
- L is 1-homogeneous in u
- Effective metric

$$g_{\mu\nu} = \frac{1}{2}\frac{\partial^2 L^2}{\partial u^\mu \partial u^\nu}$$

$F : TM \to [0, \infty)$

- $F(x, y)$ smooth on $TM \backslash \{0\}$
- F is 1-homogeneous in y
- Finsler metric (positive definite!)

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

Analytic continuation

$$ds^2 = dt^2 - dx^2$$

$$g_{\mu\nu} = diag(1, -1)$$

$$t \to t'$$

$$x \to ix'$$

$$ds^2 = d(t')^2 - d(ix')^2 = d(t')^2 + d(x')^2$$

$$g'_{\mu\nu} = diag(1, 1)$$

Kostelecký PLB 2011

SME-based Finsler norms

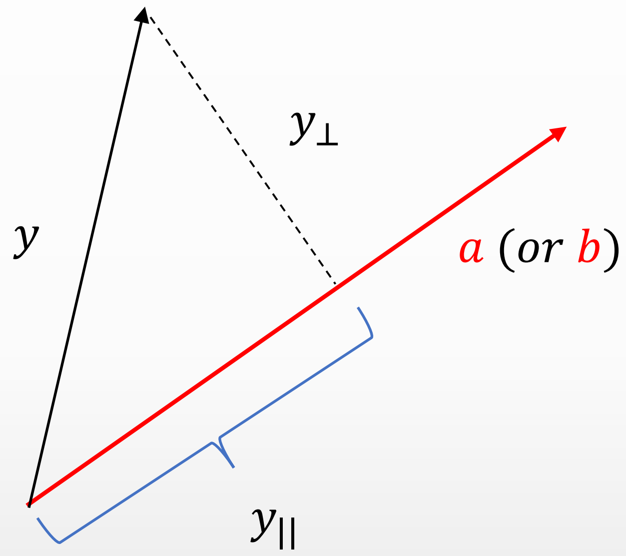
$$L_{ab} \to F_{ab} = \sqrt{y^2} + a \cdot y \pm \sqrt{b^2 y^2 - (b \cdot y)^2}$$

- Randers space

$$F_{ab}|_{b \to 0} = \sqrt{y^2} + a \cdot y = \sqrt{y^2} \pm \|a\| y_{||}$$

- b space

$$F_{ab}|_{a \to 0} = \sqrt{y^2} \pm \sqrt{b^2 y^2 - (b \cdot y)^2} = \sqrt{y^2} \pm \|b\| y_{\perp}$$



Randers 1941, Kostelecký PLB 2011

SME-based Finsler norms

- Large class of bipartite norms studied
  - includes b space and Randers space
  - includes spaces generated from H coefficients
- n-dimensional spaces considered
- Spaces categorized by isomorphism

Kostelecký, Russell, Tso PLB 2012

Map to Finsler space

$$p_\mu(u) \to (-i)^n p_j(y)$$

$$(k^{(d)})^{\alpha_1\cdots} \to i^n (k^{(d)})^{j_1\cdots}$$

$$L \to -F = -y \cdot p$$

$$u^\mu \to i^n y^j$$

- ensures nonnegative Finsler norm
- takes $\eta \to \delta$
- allows positive-definite metric

Edwards, Kostelecký PLB 2018

$$F_1 = \bar{y}\left[1 - \frac{1}{2}\sum_d \widetilde{k}^{(d)}\right]$$

$$\bar{y} = \sqrt{y^j r_{jk} y^k}$$

Finsler k spaces

Apply map to the lagrangian series expansion → Generate a Finsler norm with iterative procedure

↓

Promote to spacetime-dependent backgrounds → Takes $\tilde{k} \rightarrow \tilde{k}(x)$ and allows for curvature

Edwards, Kostelecký PLB 2018

Finsler k spaces

$$F_1 = \bar{y}\left[1 - \frac{1}{2}\sum_d \widetilde{k}^{(d)}\right]$$

$$\bar{y} = \sqrt{y^j r_{jk} y^k}$$

- Finsler norm is reversible if $F(y) = F(-y)$
  - Reversible iff $k_a = 0$
- Reversibility of $F \Rightarrow$ CPT invariance in the corresponding effective field theory for $n = 4$

Edwards, Kostelecký PLB 2018

Finsler k spaces

$$F_1^{(d_1 d_2)} = \bar{y}(1 - \tfrac{1}{2}\widetilde{k}^{(d_1)} - \tfrac{1}{2}\widetilde{k}^{(d_2)})$$

$$L_3^{(d)} \to F_3^{(d)}$$

$$F_1^{(d)} = \bar{y}(1 - \tfrac{1}{2}\widetilde{k}^{(d)})$$

- Generalizes Randers
  - generalized (α, β) metric
- Infinite set of Finsler spaces: $F_l^{(D)}$ for some subset D and order l
- Includes every perturbation of Riemann geometry that corresponds to spin-independent Lorentz violation

Finsler *k* spaces

- The Hilbert form $\omega = F_{y^i} dx^i$ is the Riemann-Finsler analogue of the n-momentum per mass

$$p_i = [1 + \tfrac{1}{2}(d - n + 1)\widetilde{k}]\hat{y}_i - \tfrac{1}{2}(d - n + 2)\widetilde{k}_i$$

$$\hat{y}^j = \frac{y^j}{\sqrt{y^k r_{km} y^m}}$$

- Rescaled in a direction-dependent way

- Momentum and velocity are generally not aligned

Edwards, Kostelecký PLB 2018

Finsler k spaces

$$
\begin{aligned}
g^{(d)}_{jk} = r_{jk} + \tfrac{1}{2}\Big[&(d-n)\widetilde{k}^{(d)}\big(r_{jk} - (d-n+2)\hat{y}_j\hat{y}_k\big) \\
&+ (d-n+2)(d-n)\big(\widetilde{k}^{(d)}{}_j\hat{y}_k + \widetilde{k}^{(d)}{}_k\hat{y}_j\big) - (d-n+1)(d-n+2)\widetilde{k}^{(d)}{}_{jk}\Big]
\end{aligned}
$$

- First-order Finsler metric
- Sufficient condition for positive-definite g places constraint on k
- Reduces to the Riemann metric for $d = n$ and $d = n - 2$
- Randers metric for $d = n - 1$

Edwards, Kostelecký PLB 2018

Characterizing the geometry

Cartan torsion vanishes → Riemann metric

Matsumoto torsion vanishes → Randers metric

Deicke 1953, Matsumoto 1974

Characterizing the geometry

Cartan torsion vanishes for $d = n$, $d = n - 2$, also for $n = 1$

$$
\begin{aligned}
C_{jkl} \quad &= \frac{1}{4\bar{y}}(d-n)(d-n+2)\sum_{(jkl)} r_{jk}\widetilde{k}^{(d)}_l - \tfrac{1}{3}(d-n+1)\widetilde{k}^{(d)}{}_{jkl} \\
&- \widetilde{k}^{(d)} r_{jk}\hat{y}_l + (d-n+1)\widetilde{k}^{(d)}{}_{jk}\hat{y}_l + \tfrac{1}{3}(d-n+4)\widetilde{k}^{(d)}\hat{y}_j\hat{y}_k\hat{y}_l - (d-n+2)\widetilde{k}^{(d)}{}_j\hat{y}_k\hat{y}_l
\end{aligned}
$$

Matsumoto torsion vanishes for $d = n - 1$

$$
\begin{aligned}
M_{jkl} = \frac{(d-n)(d-n+1)(d-n+2)}{4(n+1)\bar{y}} \sum_{(jkl)} \Big[& r_{jk}(\widetilde{k}^{(d)}\hat{y}_l - \widetilde{k}^{(d)}{}_l) \\
& + (n+1)\widetilde{k}^{(d)}{}_{jk}\hat{y}_l - \tfrac{1}{3}(n+1)\widetilde{k}^{(d)}{}_{jkl} + \tfrac{1}{3}(n-2)\widetilde{k}^{(d)}\hat{y}_j\hat{y}_k\hat{y}_l - n\widetilde{k}^{(d)}{}_j\hat{y}_k\hat{y}_l \\
& - (r_{jk} - \hat{y}_j\hat{y}_k)(\widetilde{k}^{(d)m}{}_m\hat{y}_l - \widetilde{k}^{(d)m}{}_{ml})\Big]
\end{aligned}
$$

Edwards, Kostelecký PLB 2018

Characterizing the geometry

$$F\frac{d}{d\lambda}\left(\frac{y^j}{F}\right) + G^j = 0$$

$$G^j = \tfrac{1}{2} y^m y^l g^{jk} (\partial_m g_{kl} + \partial_l g_{km} - \partial_k g_{ml})$$

$$\frac{1}{\bar{y}^2} G^{(d)j} = \widetilde{\gamma}^j{}_{\bullet\bullet} + \tfrac{1}{2}\widetilde{D}^j \widetilde{k}^{(d)} + \tfrac{1}{2}(d-n)\hat{y}^j \widetilde{D}_\bullet \widetilde{k}^{(d)} - \tfrac{1}{2}(d-n+2) r^{jm} \widetilde{D}_\bullet \widetilde{k}^{(d)}{}_m$$

Christoffel's symbol for r_{jk}

Covariant derivatives with respect to r_{jk}

Edwards, Kostelecký PLB 2018

Characterizing the geometry

- Finsler manifolds have different covariant (coordinate) bases

$$\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N^k{}_j \frac{\partial}{\partial y^k}$$

Nonlinear connection

- Connection forms (and the related curvatures) also modified

$$\Gamma^j{}_{ml} = \widetilde{\gamma}^j{}_{ml} + \tfrac{1}{4} r^{js} \Big[(d-n)\big(r_{sl} - (d-n+2)\hat{y}_s \hat{y}_l\big) \widetilde{D}_m \widetilde{k}^{(d)} \\ + (d-n+2)(d-n)\big(\hat{y}_s \widetilde{D}_m \widetilde{k}^{(d)}{}_l + (s \leftrightarrow l)\big) \\ - (d-n+2)(d-n+1) \widetilde{D}_m \widetilde{k}^{(d)}{}_{sl} + (m \leftrightarrow l) - (m \leftrightarrow s) \Big]$$

Bao, Chern, Shen An Introduction to Riemann-Finsler Geometry 2000, Edwards, Kostelecký PLB 2018

Characterizing the geometry

- Compatibility of Riemann geometry and explicit Lorentz violation

$$\widetilde{D}_j k^{(d)}{}_{a_1 \cdots a_{d-2}} = 0$$

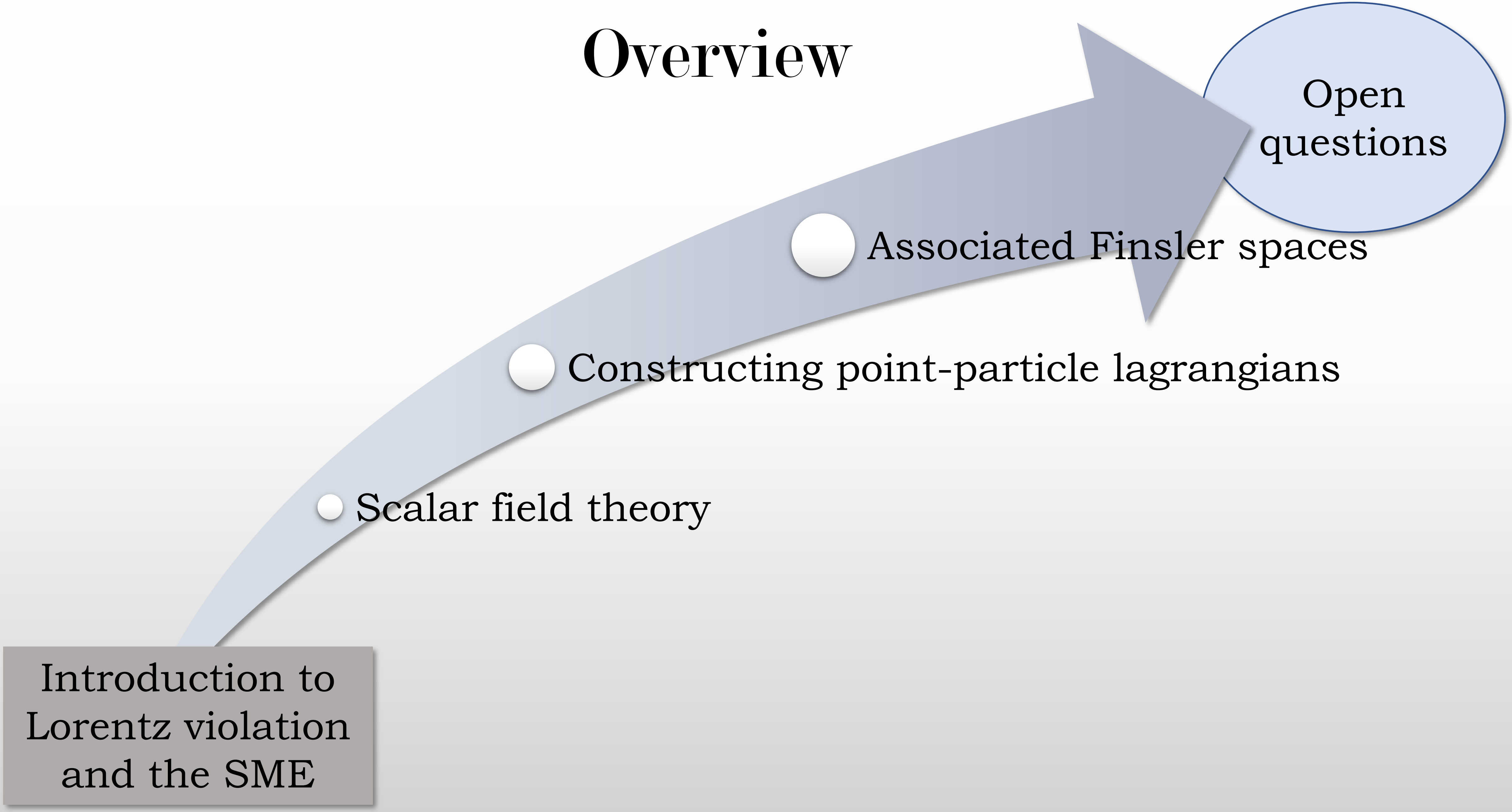
- Most quantities reduce to Riemann form for *r*-parallel backgrounds

$$\frac{1}{\bar{y}^2} G^{(d)j} \to \widetilde{\gamma}^j{}_{\bullet\bullet} \qquad\qquad \Gamma^j{}_{ml} \to \widetilde{\gamma}^j{}_{ml}$$

- Geodesics are unaffected, space is Berwald

Edwards, Kostelecký PLB 2018

Overview


Open questions
Associated Finsler spaces
Constructing point-particle lagrangians
Scalar field theory
Introduction to Lorentz violation and the SME

Open questions

Toward the future

	Quantum	Classical
Flat	$\mathcal{L} \supset \partial_\mu \phi^\dagger {k_c}^{\mu\nu} \partial_\nu \phi$	$L = -m\sqrt{u^\mu (\Omega^{-1})_{\mu\nu} u^\nu}$
Curved	$\mathcal{L} \supset e^a_\mu e^b_\nu \partial_a \phi^\dagger {k_c}^{\mu\nu} \partial_b \phi$	???

Toward the future

$$\mathcal{L} \supset \partial_\mu \phi^\dagger {k_c}^{\mu\nu} \partial_\nu \phi \Longrightarrow L = -m\sqrt{u^\mu (\Omega^{-1})_{\mu\nu} u^\nu}$$

- Can other exact lagrangians be found? What are the associated Finsler spaces?

Toward the future

	Quantum		Classical
Flat	$\mathcal{L} \supset \partial_\mu \phi^\dagger {k_c}^{\mu\nu} \partial_\nu \phi$	$\Rightarrow$	$L = -m\sqrt{u^\mu (\Omega^{-1})_{\mu\nu} u^\nu}$
	$\Downarrow$		$\Downarrow$
Curved	$\mathcal{L} \supset e^a_\mu e^b_\nu \partial_a \phi^\dagger {k_c}^{\mu\nu} \partial_b \phi$	$\Rightarrow$	???

Toward the future

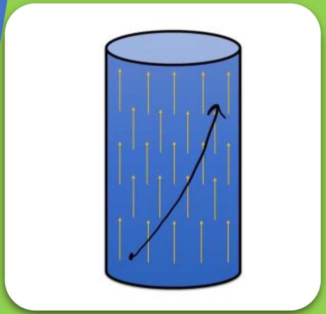
Classical

Curved

???

- How to define Lorentz-Finsler spacetimes?
- What is the geometry of curved spacetime in theories with explicit Lorentz violation?

Future interest



Conjectures supported but remain unproved

- Berwald $\Leftrightarrow$ r-parallel
- r-parallel $\Rightarrow$ usual geodesics

r-parallel backgrounds seem to be undetectable via geodesic motion

Can r-parallel components be removed from the field theory?

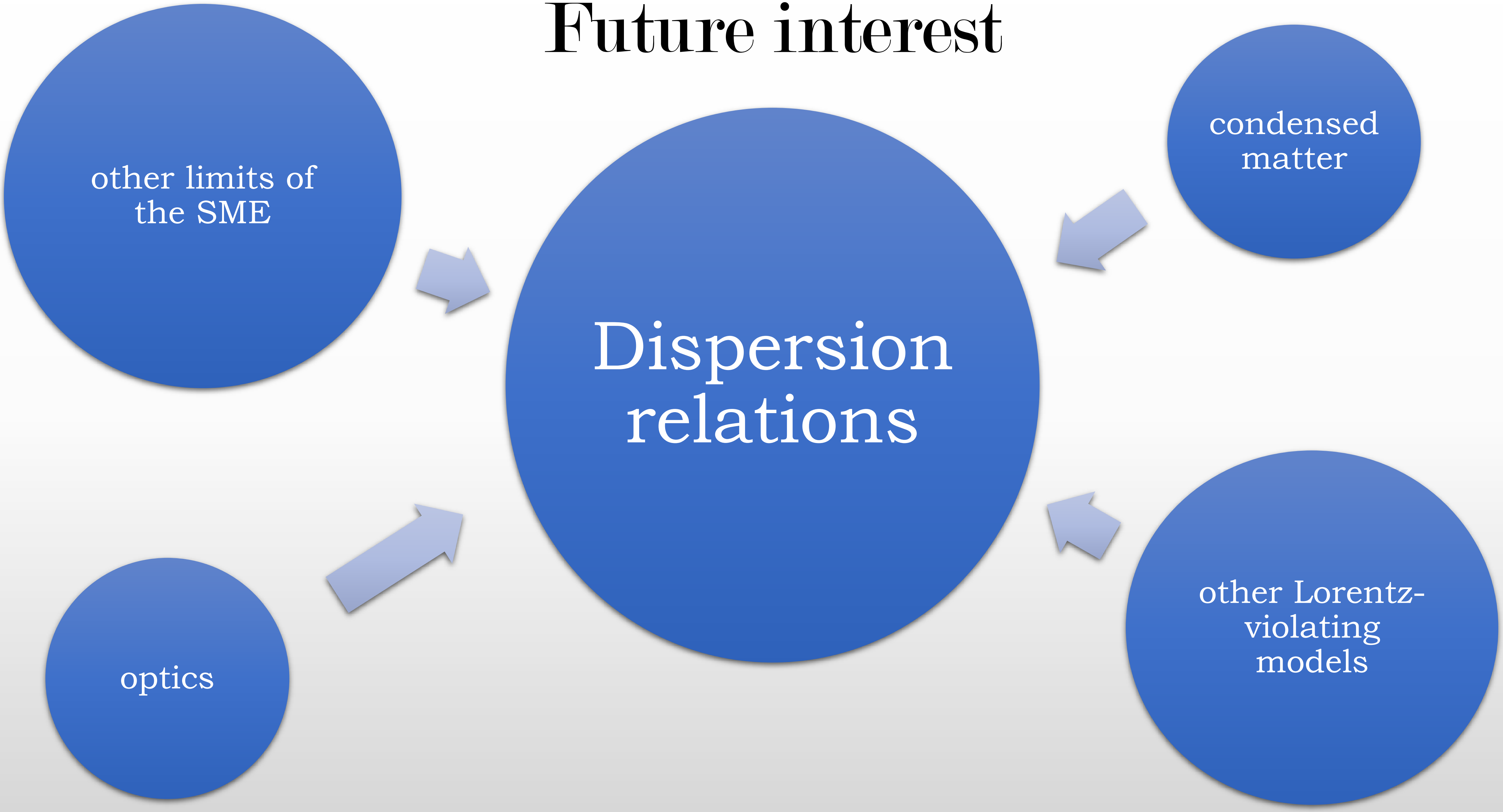
Kostelecký PLB 2011

Future interest

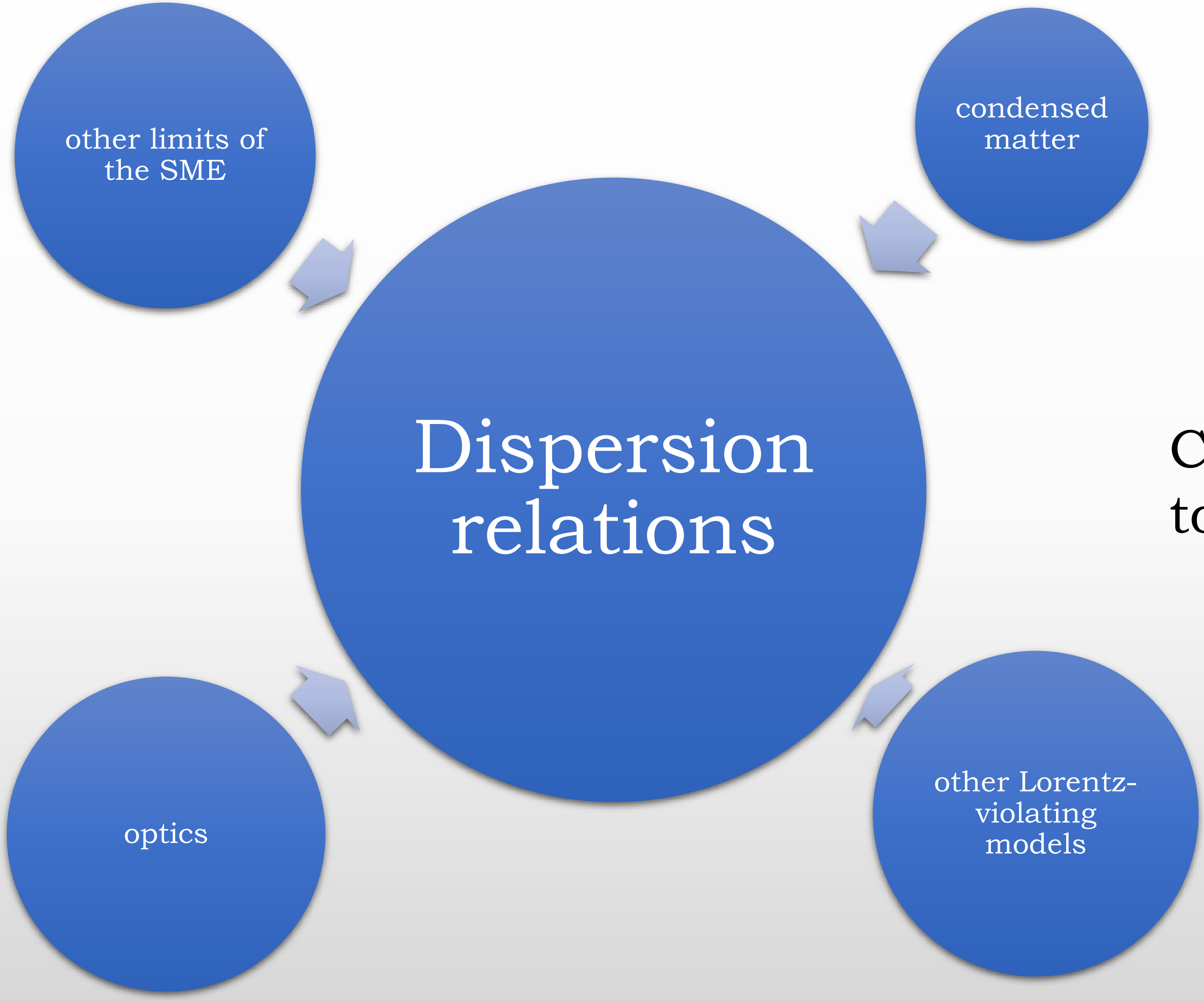
Dispersion relations

provide a connection between field theories and point particles

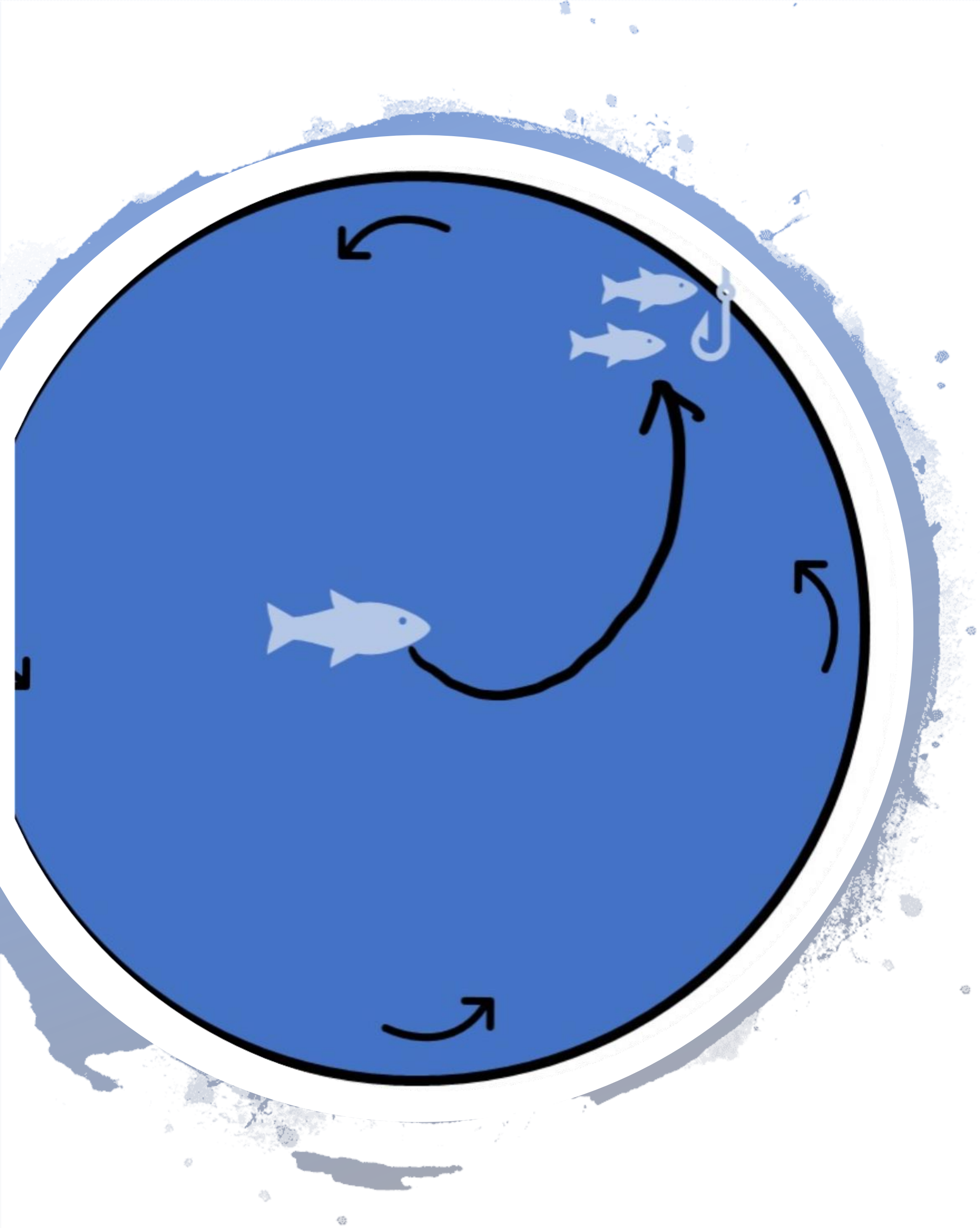
Future interest



Future interest



Can the methods here be applied to other physical systems?



Future interest: classical applications

- Classical applications of k-space?
  - Shen's fishpond (Randers)
  - Bead on a wire, magnetized chain (b space)

Shen Canad. J. Math. 2003, Foster, Lehnert PLB 2015

Future interest

Cartan torsion vanishes $\Rightarrow$ Riemann metric

$\Downarrow$

Matsumoto torsion vanishes $\Rightarrow$ Randers metric

$\Downarrow$

??? $\Rightarrow$ b space metric

k space metrics

Summary

- The general Lorentz-violating scalar field theory
- Extended method to construct point-particle lagrangians
  - any order, any mass dimension
- Associated Finsler spaces obtained
  - all perturbations of Riemann spaces describing spin-independent effects
- Many interesting open questions for future investigation