Minimal Surfaces in the Heisenberg Space

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• **J. M. Manzano**, – : Height and Area Estimates for Constant Mean Curvature Graphs in Homogeneous Space.


• –, **R. Sa Earp, E. Toubiana**: Minimal Graphs in Nil3: existence and non-existence result

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• An Open Problem
\textbf{Nil}_3(\tau)

- \textbf{Nil}_3(\tau), the Heisenberg space, is a three dimensional, simply connected Lie Group equipped with a left invariant metric.
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- It represents one of the eight Thurston geometries, that are \( \mathbb{R}^3 \), \( \mathbb{H}^3 \), \( \mathbb{S}^3 \), \( \mathbb{H}^2 \times \mathbb{R} \), \( \mathbb{S}^2 \times \mathbb{R} \), \( \text{Nil}_3 \), \( \widetilde{PSL}_2(\mathbb{R}) \), \( \text{Sol}_3 \).
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- We are able to prove the analogous of some of our results in $\text{Nil}_3(\tau)$ also in $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\text{PSL}_2(\mathbb{R})$ (by direct computation with the suitable metric or by Daniel correspondence).
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- **\text{Nil}_3(\tau)**, the Heisenberg space, is a three dimensional, simply connected Lie Group equipped with a left invariant metric.

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- We are able to prove the analogous of some of our results in \( \text{Nil}_3(\tau) \) also in \( \mathbb{R}^3 \), \( \mathbb{H}^2 \times \mathbb{R} \), \( \text{PSL}_2(\mathbb{R}) \) (by direct computation with the suitable metric or by Daniel correspondence).

- **\text{Nil}_3(\tau)\) is also known as an \( \mathbb{E}(\kappa, \tau) \) space, with \( \kappa = 0 \).
A model for $\text{Nil}_3(\tau)$ is $\mathbb{R}^3$ endowed with the Riemannian metric

$$ds^2 = (dx_1^2 + dx_2^2) + (dx_3 + \tau(x_1 dx_2 - x_2 dx_1))^2$$
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- The projection on the first two coordinates $\pi : \text{Nil}_3(\tau) \rightarrow \mathbb{R}^2$ is a Riemannian submersion with bundle curvature $\tau$. 
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- The fibers of the submersion are geodesic and coincide with the integral curves of the Killing vector field $\partial_3 = \frac{\partial}{\partial x_3}$.
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- The projection on the first two coordinates $\pi : Nil_3(\tau) :\longrightarrow \mathbb{R}^2$ is a Riemannian submersion with bundle curvature $\tau$.

- The fibers of the submersion are geodesic and coincide with the integral curves of the Killing vector field $\partial_3 = \frac{\partial}{\partial x_3}$.

- A global orthonormal frame is $E_1 = \partial_1 - \tau x_2 \partial_3$, $E_2 = \partial_2 + \tau x_1 \partial_3$, $E_3 = \partial_3$. 
Isometries in $Nil_3(\tau)$
A set of generators of the isometry group of $\text{Nil}_3(\tau)$ is

\begin{align*}
\varphi_1(x_1, x_2, x_3) &= (x_1 + c, x_2, x_3 + \tau cx_2) \\
\varphi_2(x_1, x_2, x_3) &= (x_1, x_2 + c, x_3 - \tau cx_1) \\
\varphi_3(x_1, x_2, x_3) &= (x_1, x_2, x_3 + c) \\
\varphi_4(x_1, x_2, x_3) &= ((\cos \theta)x_1 - (\sin \theta)x_2, (\sin \theta)x_1 + (\cos \theta)x_2, x_3) \\
\varphi_5(x_1, x_2, x_3) &= (x_1, -x_2, -x_3)
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ISOMETRIES IN $Nil_3(\tau)$

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- The trace of any isometry of $Nil_3(\tau)$ on the $x_1$-$x_2$ plane is an isometry of $\mathbb{R}^2$. 
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- Let $\Gamma$ be a curve in the $x_1$-$x_2$ plane. Let $\varphi$ be any isometry of $\text{Nil}_3(\tau)$.

- The curve $\varphi(\Gamma)$ is not contained in the $x_1$-$x_2$ plane in general. The projection $\pi(\varphi(\Gamma))$ of such curve on the $x_1$-$x_2$ plane is obtained from the curve $\Gamma$ by an isometry on the Euclidean $x_1$-$x_2$ plane.
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- Let $\Gamma$ be a curve in the $x_1$-$x_2$ plane. Let $\varphi$ be any isometry of $\text{Nil}_3(\tau)$.
- The curve $\varphi(\Gamma)$ is not contained in the $x_1$-$x_2$ plane in general. The projection $\pi(\varphi(\Gamma))$ of such curve on the $x_1$-$x_2$ plane is obtained from the curve $\Gamma$ by an isometry on the Euclidean $x_1$-$x_2$ plane.
- If $\Gamma$ is convex, then $\pi(\varphi(\Gamma))$ is convex, for any isometry $\varphi$ of $\text{Nil}_3(\tau)$. 
MINIMAL GRAPHS IN $\text{Nil}_3(\tau)$

Let $\tau \subset \mathbb{R}^2$. The graph of a $C^2$ function $u : \tau \to \mathbb{R}$ is a minimal surface in $\text{Nil}_3(\tau)$ if and only if $u$ satisfies the minimal surface equation:

$$2H(u) = \text{div} \, Gu + 1 + (u^2 \tau_{x_1})^2 u_{11} + 2(u + \tau_{x_2})(u^2 \tau_{x_1})u_{12} + 1 + (u + \tau_{x_2})^2 u_{22} = 0,$$

where the divergence and the norm are computed in $M(\tau)$, and $Gu$ is a vector field on $\tau$ given in coordinates by

$$Gu = ru + Z$$

where

$$Z = \tau x_2 \frac{\partial}{\partial x_1} - \tau x_1 \frac{\partial}{\partial x_2}$$

and $ru$ is the gradient of $u$ in $\mathbb{R}^2$.
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Let $\Omega \subset \mathbb{R}^2$. The graph of a $C^2$ function $u : \Omega \to \mathbb{R}$ is a minimal surface in $\text{Nil}_3(\tau)$ if and only if $u$ satisfies the minimal surface equation:

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Let $\Omega \subset \mathbb{R}^2$. The graph of a $C^2$ function $u : \Omega \rightarrow \mathbb{R}$ is a minimal surface in $\text{Nil}_3(\tau)$ if and only if $u$ satisfies the minimal surface equation:

$$2H(u) := \text{div} \left( \frac{Gu}{\sqrt{1 + \|Gu\|^2}} \right) = 0,$$

where the divergence and the norm are computed in $\mathbb{R}^2$, and $Gu$ is a vector field on $\Omega$ given in coordinates by $Gu = \nabla u + Z$ where $Z = \tau x_2 \partial_1 - \tau x_1 \partial_2$ and $\nabla u$ is the gradient of $u$ in $\mathbb{R}^2$. Developing the divergence, one gets the following equation:

$$1 + (u_2 + \tau x_1)u_11 + 2(u_1 + \tau x_2)(u_2 + \tau x_1)u_12 + 1 + (u_1 + \tau x_2)u_22 = 0.$$
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where the divergence and the norm are computed in $\mathbb{R}^2$, and $Gu$ is a vector field on $\Omega$ given in coordinates by $Gu = \nabla u + Z$ where $Z = \tau x_2 \partial_1 - \tau x_1 \partial_2$ and $\nabla u$ is the gradient of $u$ in $\mathbb{R}^2$.

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$$(1 + (u_2 - \tau x_1)^2) u_{11} - 2(u_1 + \tau x_2)(u_2 - \tau x_1) u_{12} + (1 + (u_1 + \tau x_2)^2) u_{22} = 0$$
Contents

- \( \text{Nil}(\tau) \)
- **Examples**
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- An Open Problem
EXAMPLES OF MINIMAL SURFACES IN \( \text{Nil}_3(\tau) \).

1. **Affine planes**: \( u(x_1 x_2) = ax_1 + bx_2 + c \) (vertical planes and umbrellas)

3. **Vertical catenoids**: The profile is given by a radial function \( h: \mathbb{R} \cap [-\varepsilon, \varepsilon] \to \mathbb{R} \) such that
   \[
   h'(r) = E p^1 + \tau^2 r^2 + 2 \tau^2 r^2 E^2, \\
   h(\varepsilon) = 0.
   \]

4. **Helicoids**, \( \tau = \frac{1}{2} \): \( u(x_1, x_2) = \frac{1}{2} a \arctan x_2 x_1, a > 0 \).

5. **Translationally invariant examples** (C. Figueroa, F. Mercuri, R. Pedrosa):
   \[
   c^2 \mathbb{R} u_c(x_1, x_2) = \tau x_1 x_2 + \sinh(c) 4 \tau - 2 \tau x_2 q^1 + 4 \tau^2 x_2^2 + \arcsinh(2 \tau x_2).\]

6. **Foliated examples** (B. Daniel):
   \[
   u(x_1, x_2) = x_1 f(x_2) \text{ where } f \text{ is a } C^2 \text{ function on } \mathbb{R}.
   \]
   The graphs are foliated by Euclidean straight lines (not geodesic in \( \text{Nil}_3(\tau) \) in general). They are asymptotic to \( x_3 = 0 \) on one side and to FMP's examples on the other side (\( x_2 \to \pm 1 \)).

7. **Horizontal catenoids** (B. Daniel, L. Hauswirth):
Examples of minimal surfaces in $\text{Nil}_3(\tau)$.

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3. **Helicoids, ($\tau = \frac{1}{2}$):** $u(x_1, x_2) = \frac{1}{a} \arctan \frac{x_2}{x_1}$, $a > 0$. 

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It is a one parameter, family $C_\alpha$, $\alpha > 0$, of properly embedded surfaces such that

1. $C_\alpha$ is conformally equivalent to $\mathbb{C} \setminus \{0\}$.

2. 
   \[ \pi(C_\alpha) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq \alpha \cosh \left( \frac{x_2}{\alpha} \right) \right\} \]
It is a one parameter, family $\mathcal{C}_\alpha$, $\alpha > 0$, of properly embedded surfaces such that

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2. $\pi(\mathcal{C}_\alpha) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq \alpha \cosh \left( \frac{x_2}{\alpha} \right) \right\}$

3. The projections $\pi(\mathcal{C}_\alpha)$ are obtained one from the other by a homothety and, next to the waist, they become flatter as $\alpha$ increases.
It is a one parameter, family $C_\alpha$, $\alpha > 0$, of properly embedded surfaces such that

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3. The projections $\pi(C_\alpha)$ are obtained one from the other by a homothety and, next to the waist, they become flatter as $\alpha$ increases.

4. $C_\alpha \cap \{x_2 = c\}$ is a closed embedded convex curve containing the $x_2$-axis.
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1. $C_\alpha$ is conformally equivalent to $\mathbb{C} \setminus \{0\}$.

$$\pi(C_\alpha) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \|x_1\| \leq \alpha \cosh \left(\frac{x_2}{\alpha}\right)\right\}$$

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5. $C_\alpha$ is invariant by rotation of angle $\pi$ around all the axis.
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EXISTENCE ON BOUNDED DOMAINS

For finite boundary data, on bounded domain the more general existence result is established for convex boundary and piecewise continuous boundary data. Before this result, there were existence result with more restrictive assumptions (L. Alias, M. Dajczer, J.H. De Lira, -, H. Rosenberg, R. Sa Earp, E. Toubiana).
Let $T$ be a triangle with sides $\alpha, \beta, \gamma$ and let $\varphi : \alpha \cup \beta \to \mathbb{R}$ a continuous function.
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- R. Sa Earp, E. Toubiana proved the existence of a unique minimal extension of $\varphi$ over $T$ assuming the value $\infty$ in the interior of $\gamma$. 

For $\alpha < \pi$, S. Cartier proved existence of minimal graphs on $W_\alpha$ with zero boundary value and with linear growth (deformation of an umbrella).

For $\pi - \alpha < \theta < \pi$, R. Sa Earp, E. Toubiana proved existence of minimal graphs on $W_\alpha$ with zero boundary value and with at least quadratic growth.

These results are in contrast with the $\mathbb{R}^3$ case, where a minimal solution with zero boundary value on a wedge of angle $\pi$ is zero (H. Rosenberg, R. Sa Earp).
Let $T$ be a triangle with sides $\alpha, \beta, \gamma$ and let $\varphi : \alpha \cup \beta \to \mathbb{R}$ a continuous function.

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Scherk Type Surfaces

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**Scherk Type Surfaces**

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These results are in contrast with the $\mathbb{R}^3$ case, where a minimal solutions with zero boundary value on a wedge of angle $< \pi$ is zero (H.Rosenberg, R. Sa Earp).
Theorem (, R. Sa Earp, E. Toubiana)

Let $\Omega \subset \mathbb{R}^2$ be an unbounded, convex domain different from an half-plane. Let $\varphi$ be a continuous function on $\Gamma = \partial \Omega$ except at a discrete set of points where $\varphi$ has left and right limit. Then there exists a minimal extension $u$ of $\varphi$ over $\bar{\Omega}$. Moreover the boundary of the graph of $u$ contains the vertical segments between the left and the right limits of $\varphi$ at the discontinuity points.
Graphs on unbounded domains

Theorem (, R. Sa Earp, E. Toubiana)

- Let $\Omega \subset \mathbb{R}^2$ be an unbounded, convex domain different from a half-plane. Let $\varphi$ be a continuous function on $\Gamma = \partial \Omega$ except at a discrete set of points where $\varphi$ has left and right limit. Then there exists a minimal extension $u$ of $\varphi$ over $\bar{\Omega}$. Moreover the boundary of the graph of $u$ contains the vertical segments between the left and the right limits of $\varphi$ at the discontinuity points.

- Let $\Omega$ be a half-plane and let $\Gamma = \partial \Omega$. Let $\varphi$ be a bounded function on $\Gamma$, continuous except at a discrete set of points where $\varphi$ has left and right limit. Then there exists a 1-parameter family of minimal extensions $u$ of $\varphi$ over $\bar{\Omega}$. Moreover the boundary of the graph of $u$ contains the vertical segments between the left and the right limits of $\varphi$ at the discontinuity points.
Theorem ( - , R. Sa Earp, E. Toubiana)
Main Steps and Tools of the Proof

1. \( \Omega_n \) relatively compact domains exhausting \( \Omega \). Choose boundary data \( \varphi_n \) on each \( \partial \Omega_n \). Then solve the Dirichlet problem on \( \Omega_n \) : solutions \( u_n \).
Main Steps and Tools of the Proof

1. $\Omega_n$ relatively compact domains exhausting $\Omega$. Choose boundary data $\varphi_n$ on each $\partial \Omega_n$. Then solve the Dirichlet problem on $\Omega_n$: solutions $u_n$.

2. Prove that there is a subsequence of $u_n$ converging to a minimal solution $u$ on $\Omega$ on any compact subset of $\Omega$. 

Schauder: $C^1 \Rightarrow C^2$.

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Theorem (H. Rosenberg, R. Souam, E. Toubiana) Let $\Omega \subset \mathbb{R}^2$ be a relatively compact domain and let $u : \Omega \rightarrow \mathbb{R}$ satisfy the minimal surface equation. Then, for any positive constant $C_1$, $C_2$, there exists a constant $\alpha = \alpha(C_1, C_2, \Omega)$ such that for any $p \in \Omega$ with $d(p, \partial \Omega) \geq C_2$ and $|u| < C_1$ on $\Omega$, we have

$$|\nabla u(p)| < \alpha$$

for any $p \in \Omega$. Uniform height estimates implies convergence. Then one has to use barrier in order to prove that boundary data are right.
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- **Uniform height estimates implies convergence**

- Then one has to use barrier in order to prove that boundary data are right.
GEOMETRIC IDEA OF THE PROOF IN THE HALFPLANE CASE

Assume that the half-plane is \( \mathcal{I} = \{ (x_1, x_2) \in \mathbb{R}^2 | x_2 > d \} \).

Consider \( x_3 = ax_2 + b \), where \( a > 0 \), \( b > \sup \{ x_2 = d \} \).

For any \( n \in \mathbb{N} \), consider the strip \( \mathcal{I}_n = \{ (x_1, x_2) \in \mathbb{R}^2 | d < x_2 < n \} \) and the continuous function \( \gamma_n(p) = \gamma(p) \), \( p = (x_1, d) + (a, b) \), \( p = (x_1, n) \).

Exhaust \( \mathcal{I}_n \) by rectangles \( R_{n, k} \), \( k = \{ (x_1, x_2) \in \mathbb{R}^2 | |x_1| < k, d < x_2 < n \} \) and define \( \gamma_{n, k}(p) = \gamma_{n}(p) \) for \( p \in \partial \mathcal{I}_n \setminus \partial R_{n, k} \), and it is monotone on the vertical sides of \( R_{n, k} \).

\{ (x_1, x_2) \in \bar{R}_{n, k}, x_1 = \pm k \}.
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We solve the Dirichlet problem on $\mathcal{R}_{n,k}$ with boundary values equal to $\varphi_{n,k}$. Let $\{u_{n,k}\}_{k \in \mathbb{N}}$ be the solution.

By the compactness theorem, for $k \to 1$, $u_{n,k}$ converges to a solution $u_{n}$ defined on $\mathcal{R}_{n}$. Moreover each $u_{n}$ is between two planes. Then we let $n \to 1$ and we get a subsequence of $u_{n}$ converging to a solution on the halfplane. One proves that all the solutions $u_{n}$ and $u_{n,k}$ take the right boundary value at continuity points using the technique of barriers, and at the discontinuity points, by hand.
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- The existence of one parameter family of solutions is achieved by changing the slope of the initial plane that one uses as supersolution.
A non existence result

Theorem (-, R. Sa Earp, E. Toubiana)
Let $\Omega$ be a domain such that $\Gamma = \partial \Omega$ is a non convex $C^k$ curve, $k \geq 0$. If either:

1. $\Omega$ is bounded,
2. $\Omega$ is unbounded and contained either in a wedge or in a strip.

Then there exists a $C^k$ function on $\bar{\Omega}$ that does not admit a minimal extension over $\bar{\Omega}$.

Our proof holds in $\mathbb{R}^3$ as well. In $\mathbb{R}^3$, for $\Omega$ bounded, the analogous statement is by R. Finn (continuous boundary data), H. Jenkins and J. Serrin ($C^2$ boundary data, with arbitrary small absolute value); for $\Omega$ unbounded we did not find a statement in the literature.

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Contents

• Nil(τ)
• Examples
• Existence
• Area Growth
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TWO TYPES OF AREA GROWTH OF A GRAPH $G$ IN $\text{Nil}_3(\tau)$

Let $p_0 \in G$ and let $B(p_0, R)$ be a geodesic ball in $\text{Nil}_3(\tau)$ of radius $R$ centered at $p_0$. Let $\alpha > 0$ and assume

$$\liminf_{R \to \infty} \frac{\text{Area}(G \cap B(p_0, R))}{R^\alpha} > 0, \quad \left( \limsup_{R \to \infty} \frac{\text{Area}(G \cap B(p_0, R))}{R^\alpha} < \infty \right)$$
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Then we say that $G$ has extrinsic area growth of order at least (at most) $\alpha$. 

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- Let $D(x_0, R)$ a disk in $\mathbb{R}^2$, and $C(x_0, R) = \pi^{-1}(D(x_0, R))$ the cylinder above it. Let $\alpha > 0$ and assume

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Then we say that $G$ has cylindrical area growth of order at least (at most) $\alpha$. 

AREA GROWTH OF SOME ENTIRE GRAPHS
Area growth of some entire graphs

- **Umbrellas**: $u(x_1, x_2) = ax_1 + bx_2$. Extrinsic (and cylindrical): cubic. Intrinsic: cubic. Conformal type: hyperbolic.
**Area Growth of Some Entire Graphs**

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Geodesic balls in $\text{Nil}_3(\tau)$

**Lemma (M. Manzano, -)**

Given $R > 0$, let $B_R(0)$ be the geodesic ball in $\text{Nil}_3(\tau)$ centered at the origin and let $D_R(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < R^2\}$.

- If $R \leq \frac{\pi}{2\tau}$, then $B_R(0) \subset D_R(0) \times ] - R, R[.$
- If $R > \frac{\pi}{2\tau}$, then $B_R(0) \subset D_R(0) \times ] - \frac{\pi^2 + 4\tau^2 R^2}{4\tau\pi}, \frac{\pi^2 + 4\tau^2 R^2}{4\tau\pi}[.$

For the proof, we write the explicit equations of the geodesics in $\text{Nil}_3(\tau)$ and we estimates their maximal height in the ball $B_R(0)$.

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Estimates on Area Growth

Theorem (M. Manzano, -)

Let $\mathcal{M} \to \mathrm{Nil}$ be a minimal entire graph. Then $\mathcal{M}$ has at least cubic and at most quartic cylindrical area growth.

Let $\mathcal{M} \to \mathrm{Nil}$ be an entire minimal graph. Then $\mathcal{M}$ has at least quadratic and at most cubic extrinsic area growth.

The at least part in (C) is based on area $\left( \mathcal{M} \setminus C_R(x_0) \right)$ area $\left( \mathcal{M}_0 \setminus C_R(x_0) \right)$. Where $\mathcal{M}_0$ is the horizontal umbrella centered at $p_0$ such that $\mathcal{M}(p_0) = x_0$.

The at most in (E) part comes form area $\left( \mathcal{M} \setminus B_R(0) \right) \subseteq \mathcal{M}_0 \setminus C_R(x_0)$. Length $\mathcal{M}$ is a graph over $\mathcal{M}_0$ by a function $u$, $\mathcal{M}_0 \setminus C_R(x_0) \subseteq \mathcal{M}_0 \setminus D_R(0) \subseteq \mathcal{M}_0 \setminus D_R(0)$ for some positive function $h(x_0)$. Where $\mathcal{M}_0$ is a graph over $\mathcal{M}_0$ by a function $u$, $\mathcal{M}_0 \setminus C_R(x_0) \subseteq \mathcal{M}_0 \setminus \mathcal{M}_0$. Length $\mathcal{M}$ is a graph over $\mathcal{M}_0$ by a function $u$, $\mathcal{M}_0 \setminus C_R(x_0) \subseteq \mathcal{M}_0 \setminus D_R(0)$ for some positive function $h(x_0)$. Where $\mathcal{M}_0$ is a graph over $\mathcal{M}_0$ by a function $u$, $\mathcal{M}_0 \setminus C_R(x_0) \subseteq \mathcal{M}_0 \setminus D_R(0)$ for some positive function $h(x_0)$. Where $\mathcal{M}_0$ is a graph over $\mathcal{M}_0$ by a function $u$, $\mathcal{M}_0 \setminus C_R(x_0) \subseteq \mathcal{M}_0 \setminus D_R(0)$ for some positive function $h(x_0)$. Where $\mathcal{M}_0$ is a graph over $\mathcal{M}_0$ by a function $u$, $\mathcal{M}_0 \setminus C_R(x_0) \subseteq \mathcal{M}_0 \setminus D_R(0)$ for some positive function $h(x_0)$. Where $\mathcal{M}_0$ is a graph over $\mathcal{M}_0$ by a function $u$, $\mathcal{M}_0 \setminus C_R(x_0) \subseteq \mathcal{M}_0 \setminus D_R(0)$ for some positive function $h(x_0)$.
Estimates on Area Growth

Theorem (M. Manzano, -)

(C) Let $\Sigma \subset \text{Nil}_3$ be a minimal entire graph. Then $\Sigma$ has at least cubic and at most quartic cylindrical area growth.
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- The at most in (E) part comes from
  \[
  \text{area}(\Sigma \cap B_R(0)) \leq \int_{\Omega(R)} (1 + |Z|) + h(R)\text{length}(\partial\Omega(R)).
  \]
  where $\Sigma$ is a graph over $\Omega$ by a function $u$, $\Omega(R) = \Omega \cap D_R(0)$ and $B_R(0) \subset D_R(0) \times [-h(R), h(R)]$ for some positive function $h$. 
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As a byproduct of the previous inequality: in $\mathbb{R}^3$, the intrinsic area growth is at most quadratic.
Contents

• Nil(τ)

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**Theorem (Height Growth)**

(Manzano, -) Let $\Sigma$ be an entire minimal graph in $\text{Nil}_3(\tau)$, given by a function $u \in C^\infty(\mathbb{R})$ and let $r = \sqrt{x_1^2 + x_2^2}$. Then
THEOREM (HEIGHT GROWTH)
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1. There exists $B > 0$ such that
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As far as we know, there is no example with more than quadratic height growth.
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Conjecture

The height growth of an entire minimal graph in $\text{Nil}_3(\tau)$ is at most quadratic.
Sketch of the proof of the Height Growth

The proof relies on a gradient estimate for entire space-like graphs in Lorentz-Minkowski space $L^3$, with constant mean curvature, related to our graphs by the Calabi-type correspondence (M. Manzano, H. Lee).

The Calabi-type correspondence implies that there exists $v_2 \in C^1(\mathbb{R}^2)$ such that the graph of $v$ is a space-like surface in $L^3$ with constant mean curvature $\tau$. Moreover, $v$ satisfies

$$(1 + |\nabla v|^2) = 1 (\nabla v \in \mathbb{R}^2).$$

Using the properties of space-like constant mean curvature surfaces in $L^3$, one can prove that there exists a constant $A > 0$ such that

$$|\nabla v|^2 \leq A (1 + \nabla^2 v)^2.$$ 

This yields that $1 + |\nabla v|^2 \leq A (1 + \nabla^2 v)^2$ and this easily gives that $|\nabla v| \leq B (1 + \nabla^2 v)$. The height growth easily follows.
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- The Calabi-type correspondence implies that there exists $v \in C^\infty(\mathbb{R}^2)$ such that the graph of $v$ is a space-like surface in $\mathbb{L}^3$ with constant mean curvature $\tau$. Moreover $v$ satisfies $(1 - |\nabla v|^2)(1 + |Gu|^2) = 1$ ( $\nabla v$ in $\mathbb{R}^2$).
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- Using the properties of space-like constant mean curvature surfaces in $\mathbb{L}^3$, one can prove that the exists a constant $A > 0$ such that $|\nabla v|^2 \leq 1 - \frac{A}{(1+r^2)^2}$.

- This yields that $1 + |Gu|^2 \leq A^{-1}(1 + r^2)^2$ and this easily gives that $|Gu| \leq B(1 + r^2)$. The height growth easily follows.
Sketch of the proof that an entire minimal graph \( \Sigma \) in \( \text{Nil}_3(\tau) \) has at least quadratic extrinsic area growth.

One has

\[
\text{area}(\Sigma \setminus B(0)) \preceq \text{area}(\Sigma \setminus D(0) \times \mathbb{R}) \sim \text{area}(\Sigma \setminus D(0))^2 \:
\]

The distance in \( \text{Nil}_3(\tau) \) from the origin to \( p = (x_1, x_2, x_3) \) is equivalent to

\[
p^2 = \max \{ q x_1^2 + x_2^2, 1 \} = \max \{ q x_1^2 + x_2^2, 1 \}
\]

As \( |u| \prec C(1 + r)^3 \) on \( D \setminus D(0)^3 \) and \( C(1 + r)^4 \prec 2^{CR^2} \):
Sketch of the proof that an entire minimal graph $\Sigma$ in $\text{Nil}_3(\tau)$ has at least quadratic extrinsic area growth.

One has

$$\text{area}(\Sigma \cap B_R(0)) \sim \text{area}(\Sigma \cap D_R(0) \times] - 2CR^2, 2CR^2[)$$
$$\geq \text{area}(\Sigma \cap D_{R^3}(0) \times] - 2CR^2, 2CR^2[)$$

(the distance in $\text{Nil}_3(\tau)$ from the origin to $p = (x_1, x_2, x_3)$ is equivalent to $\delta_{\sqrt{2C}}(p) = \max\{\sqrt{x_1^2 + x_2^2}, \frac{1}{\sqrt{2C}} \sqrt{|x_3|}\}$.)
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  \[
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  But one knows that the cylindrical area growth is at least as \( (R^3_{\frac{3}{2}})^3 \), that gives the desired estimate.
Recall the

**Conjecture**

The height growth of an entire minimal graph in $\text{Nil}_3(\tau)$ is at most quadratic.
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The height growth of an entire minimal graph in $\text{Nil}_3(\tau)$ is at most quadratic.

If the conjecture is true, then the extrinsic area growth of an entire minimal graph in $\text{Nil}_3(\tau)$ would be cubic.
**Theorem (Collin-Krust Type Result)**

(Manzano, -)

Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $u \in C^\infty(\Omega)$ be a solution in $\Omega$ of the minimal surface equation in $\text{Nil}_3(\tau)$, such that $u|_{\partial \Omega} = 0$. Denote $M(r) = \sup_{\rho \leq r} |u|$, then

$$\limsup_{r \to \infty} \frac{M(r)}{r} > 0.$$
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Let $A = \{x \in \Omega : u(x) > 0\}$ and $\Lambda_r = \{x \in A : \rho(x) = r\}$. If there exists a positive constant $C$ such that $\text{Length}(\Lambda_r) < C$, then

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THEOREM (COLLIN-KRUST TYPE RESULT)

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$$\limsup_{r \to \infty} \frac{M(r)}{r^2} > 0$$

The result is sharp because of the plane and the catenoid (there is a previous non sharp result in $\text{Nil}_3(\tau)$ by C. Leandro and H. Rosenberg).
Contents

• \( \text{Nil}(\tau) \)

• Examples

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• An Open Problem
AN OPEN PROBLEM (STRONG HALF-SPACE THEOREM)

Two properly immersed minimal surfaces in \( \text{Nil}^3 \) that do not intersect are either two parallel vertical planes or an entire minimal graph and its image by a vertical translation. The conjecture is proved provided one of the surfaces is either a graph or a vertical plane (Daniel-Meeks-Rosenberg (2011), Daniel-Hauswirth (2009)). On the way to prove the conjecture, one is led to study complete stable minimal surfaces in \( \text{Nil}^3 (\varepsilon) \). M. Manzano, J. Perez, M. Rodriguez (2011) classified complete stable immersed minimal surfaces, provided the surfaces are parabolic (and so the conjecture is proved in this case). S. Y. Cheng, S.T. Yau (1975) proved that if a surface has quadratic (intrinsic) area growth, then it is parabolic. We showed many examples with intrinsic cubic area growth.
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**Theorem (M. Manzano, -)**

Let $\Sigma$ be a minimal stable surface in $Nil_3(\tau)$. If the angle function $\nu = \langle E_3, N \rangle$ is such that $\nu^2 \in L^1(\Sigma)$, then $\Sigma$ is a vertical plane.
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The proof depends on the gradient estimate that we got before that prevents $\Sigma$ to be a graph when $\nu^2 \in L^1(\Sigma)$. Then we use a classification theorem by J. M. Espinar (2013).
Gracias
## SUMMARY

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<td>$H = 0$</td>
<td>$\leq \mathbb{R}^{R\sqrt{-\kappa}}$</td>
<td>$\geq e^{R\sqrt{-\kappa}}$</td>
<td>$\leq \mathbb{R}^{R\sqrt{-\kappa}}$</td>
<td></td>
</tr>
<tr>
<td>Graphs with</td>
<td>$H = 0$</td>
<td>$\mathbb{E}(\kappa, \tau)$</td>
<td>$\mathbb{R}^3$</td>
<td>$\mathbb{R}^3$</td>
<td>$\mathbb{R}^3$</td>
<td></td>
</tr>
<tr>
<td>zero boundary</td>
<td></td>
<td>$\kappa &lt; 0$</td>
<td>$\leq R^2$</td>
<td>$\leq R^2$</td>
<td>$\leq R^2$</td>
<td></td>
</tr>
<tr>
<td>values</td>
<td></td>
<td>$\mathbb{E}(\kappa, \tau)$</td>
<td>$\leq R^3$</td>
<td>$\leq R^3$</td>
<td>$\leq R^3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\kappa &lt; 0$</td>
<td>$\leq \mathbb{R}^{R\sqrt{-\kappa}}$</td>
<td>$\leq \mathbb{R}^{R\sqrt{-\kappa}}$</td>
<td>$\leq \mathbb{R}^{R\sqrt{-\kappa}}$</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** EAG=Extrinsic Area Growth, CAG=Cylindrical Area Growth, IAG=Intrinsic Area Growth, CT=Conformal Type