Jenkins-Serrin problem for translating graphs

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May 22, 2019

Based on joint works with Eddygledson S. Gama (UFC), Jorge de Lira (UFC), and Francisco Martín (UGR).

Let M^n be a Riemannian manifold and $\Omega \subset M$ be a domain with piecewise smooth boundary. Assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where the sets Γ_i are disconnected so that any smooth connected component of Γ_i does not intersect any another smooth connected component of Γ_j for $i, j \in \{0, 1, 2\}$.

A classical problem is to find the sufficient and necessary conditions for the solvability of the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x), & \text{in } \Omega; \\ u = c, & \text{on } \Gamma_0; \\ u = +\infty, & \text{on } \Gamma_1; \\ u = -\infty, & \text{on } \Gamma_2, \end{cases} \tag{1}$$

where $H\colon M \to \mathbb{R}$ is a Lipschitz function and $c\colon \Gamma_0 \to \mathbb{R}$ is a continuous function.

The most famous example of solutions of (1) in $\mathbb{R}^2$ with $\Omega = [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ was given by H. Scherk in 1834. Namely, he proved that the function

$$u = \log(\cos x / \cos y)$$

is a solution of (1) with $\Gamma_0 = \emptyset$ and $H \equiv 0$, obtaining the Scherk's minimal surface.

More than a hundred years later H. Jenkins and J. Serrin found the necessary and sufficient conditions for the existence of solutions of (1) in $\mathbb{R}^2$ with $H \equiv 0$. They related the existence of solutions of (1) with conditions involving the length of "admissible polygons" inside the domain.

Now the Dirichlet problem (1) is known as the Jenkins-Serrin problem.

Theorem 1 (Jenkins-Serrin)

Let $\Omega \subset \mathbb{R}^2$ be as above and assume also that no two arcs A_i and no two arcs B_i have a common endpoint. For continuous $f_i \colon C_i \to \mathbb{R}$, there exists a minimal Jenkins-Serrin solution $u \colon \Omega \to \mathbb{R}$ with $u|_{C_i} = f_i$ if and only if

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}) \text{ and } 2\beta(\mathcal{P}) < \ell(\mathcal{P}).$$

for any admissible polygon $\mathcal{P}$ ($\alpha(\mathcal{P}) = \sum_{A_i \subset \mathcal{P}} |A_i|$, $\beta(\mathcal{P}) = \sum_{B_i \subset \mathcal{P}} |B_i|$). If $\{C_i\} = \varnothing$, we require also $\alpha(\partial\Omega) = \beta(\partial\Omega)$ for $\mathcal{P} = \partial\Omega$. If $\{C_i\} \neq \varnothing$, u is unique, and if $\{C_i\} = \varnothing$, u is unique up to adding a constant.

The existence of minimal solutions imposes restrictions to the domain.

Theorem 2 (Jenkins-Serrin)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $u \colon \Omega \to \mathbb{R}$ a solution of the minimal graph equation satisfying $u \to \pm\infty$ as $x \to \gamma$, where γ is a smooth connected component of $\partial\Omega$. Then γ is a geodesic.

After Jenkins and Serrin this problem has been considered in many different settings by many different researchers (among others):

- J. Spruck for CMC surfaces in $\mathbb{R}^3$
- B. Nelli and H. Rosenberg for minimal case in $\mathbb{H}^2 \times \mathbb{R}$
- A.L. Pinheiro for minimal case in $M^2 \times \mathbb{R}$
- P. Collin and H. Rosenberg on unbounded domains in $\mathbb{H}^2$
- L. Mazet, M. Rodríguez and H. Rosenberg on unbounded domains in $\mathbb{H}^2$
- J. Gálvez and H. Rosenberg on unbounded domains in M^2 with negative constant upperbound for curvature
- M. Eichmair and J. Metzger for CMC case and Jang equation in M^n, $2 \leq n \leq 7$
- M.H. Nguyen for minimal case in Sol_3
- P. Klaser and A. Menezes for CMC case in Sol_3

Translating graphs

A hypersurface $\Sigma \subset M \times \mathbb{R}$ is a translating soliton with respect to the parallel vector field $X = \partial_t$ (with translation speed $c \in \mathbb{R}$) if

$$\mathbf{H} = c\, X^{\perp},$$

where $\mathbf{H}$ is the mean curvature vector field of Σ and $\perp$ indicates the projection onto the normal bundle of Σ.

In particular, if N is a normal vector field along Σ, then we have

$$H = c\langle X, N\rangle, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian product metric in $M \times \mathbb{R}$.

A translating soliton can be described locally in non-parametric terms as a graph

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

of a smooth function u defined in a domain $\Omega \subset M$. In this case, we denote $\Sigma = \mathrm{Graph}[u]$ and we refer to those solitons as *translating graphs*.

From (2) we get that u satisfies

$$\operatorname{div}\left(\frac{\nabla u}{W}\right) = \frac{c}{W}, \qquad W = \sqrt{1 + |\nabla u|^2}. \tag{3}$$

In this case, Σ can be oriented by the normal vector field

$$N = \frac{1}{W}(X - \nabla u).$$

Some examples of translating solitons:

- Grim reaper curve Γ in $\mathbb{R}^2$, given by $f\colon (-\pi/2, \pi/2) \to \mathbb{R}^2$

$$f(x) = (x, -\log\cos x);$$

- Grim reaper hyperplane $\Gamma \times \mathbb{R}^{n-1}$;
- Tilted grim reapers;
- Bowl soliton (the only convex translator that is an entire graph (Wang), the only entire graph over $\mathbb{R}^2$ (Spruck-Xiao));
- Translating catenoid (”wing-like” soliton);
- Vertical plane;
- Δ-wing (asymptotic to tilted grim reapers);
- Scherk-like translators.

Lemma 3 (T. Ilmanen)

Translating solitons with translation speed $c \in \mathbb{R}$ are minimal hypersurfaces in $M \times \mathbb{R}$ with respect to the Ilmanen's metric $g_c = e^{\frac{2c}{m}t}(\sigma + \mathrm{d}t^2)$.

Lemma 4

All translating graphs are stable in $M \times \mathbb{R}$ endowed with Ilmanen's metric g_c.

Remark

This lemma was proved by Shahriyari for vertical translating graphs in $\mathbb{R}^3$ and by Zhou for any translating graph in $M^2 \times \mathbb{R}$, where M is a Riemannian surface. We extend the result to all dimensions.

We actually have that translating graphs over $\Omega \subset M$ are area minimizing in $(\Omega \times \mathbb{R}, g_c)$.

Definition 5 (Nitsche curve)

Let $\Omega \subset M$ be a domain and $\Gamma \subset M \times \mathbb{R}$ a Jordan curve. Γ is called a Nitsche curve, if it admits a parametrization

$$\Gamma(t) = \{(\alpha(t), \beta(t)) \colon t \in \mathbb{S}^1\}$$

s.t. $\alpha(t)$ is a monotone parametrization of $\partial\Omega$. This means that $\alpha \colon \mathbb{S}^1 \to \partial\Omega$ is continuous and monotone, and there exist closed disjoint intervals $J_1, \ldots, J_v$ such that $\alpha|_{J_i}$ is constant for all i and $\alpha|_{\mathbb{S}^1 \setminus \cup J_i}$ is one-to-one and smooth.

Definition 6 (Admissible domain)

Let $\Omega \subset M$ be a domain. Then Ω is admissible if it is geodesically convex and $\partial\Omega$ is a union of geodesic arcs $A_1, \ldots, A_s, B_1 \ldots, B_r$, convex arcs $C_1, \ldots, C_t$, the end points of these arcs and that no two arcs A_i and no two arcs B_i have a common endpoint.

Definition 7 (Admissible polygon)

Let Ω be an admissible domain. Then $\mathcal{P}$ is an admissible polygon if $\mathcal{P} \subset \Omega$ and the vertices of $\mathcal{P}$ are chosen among the vertices of Ω.

Let Γ be a Nitsche curve over the boundary $\partial\Omega$ of an admissible domain Ω. By a translating soliton with boundary Γ we mean a translating soliton in $\Omega \times \mathbb{R}$ that is a graph over Ω.

Using classical results about the solvability of the Plateau problem, we can prove that any Nitsche curve over an admissible domain admits a unique translating soliton with it as the boundary.

Theorem 8

Let Ω be an admissible domain in M and Γ a Nitsche curve over $\partial\Omega$. Then there exists a unique translating soliton with boundary Γ.

Idea of the proof:

Since the boundary $\partial\Omega$ consists of geodesic and convex arcs, the boundary $\partial(\Omega \times \mathbb{R})$ is mean convex, and it remains mean convex also after changing to the Ilmanen's metric.

Therefore we can apply solvability results of the Plateau's problem (Meeks-Yau, Morrey) and find an embedded minimal (w.r.t. g_c) disk $\Sigma \subset \Omega \times \mathbb{R}$ with boundary Γ.

It remains to prove that $\text{int}(\Sigma)$ is a graph over Ω, but this follows by somewhat standard arguments for minimal surfaces.

In order to prove a Jenkins-Serrin type result for the translating graphs, we will use minimal surfaces as barriers. For this we need the following maximum principle.

Lemma 9 (Maximum principle)

Let $\Omega \subset M$ be an admissible domain. Suppose that u_1 and u_2 satisfy

$$div\left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}}\right) \geq div\left(\frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\right),$$

and $\liminf(u_2 - u_1) \geq 0$ *for any approach of* $\partial\Omega$*, with possible exception of finite numbers of points* $\{q_1, \ldots, q_r\} =: E \subset \partial\Omega$*. Then* $u_2 \geq u_1$ *on* $\partial\Omega \setminus E$ *with strict inequality unless* $u_2 = u_1$*.*

The proof is a modification of a similar result for CMC surfaces by J. Spruck.

Theorem 10 (Gama-H-Lira-Martín)

Let $\Omega \subset M$ be an admissible domain with $\{B_i\} = \emptyset$. Given any continuous boundary data $f_i \colon C_i \to \mathbb{R}$, there exists a Jenkins-Serrin solution $u \colon \Omega \to \mathbb{R}$ for the translating soliton equation with $u|_{C_i} = f_i$, if for any admissible polygon $\mathcal{P}$ we have

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}). \tag{4}$$

Proof:

Define a Nitsche curve $\Gamma_n = (\alpha_n, \beta_n)$ by setting $\beta_n = n$ on $\{A_i\}$ and $\beta_n = \min\{f_i, n\}$ on C_i for all i.

By Theorem 8, for all $n \in \mathbb{N}$, there exists $u_n \colon \Omega \to \mathbb{R}$ so that Graph$[u_n]$ is a translating soliton in $\Omega \times \mathbb{R}$ with boundary Γ_n, and by the comparison principle, we also have that $\{u_n\}$ is a monotone sequence.

By Pinheiro's results, (4) guarantees that there exists a Jenkins-Serrin solution $v\colon \Omega \to \mathbb{R}$ of the minimal graph equation with continuous boundary data f_i.
Since

$$\operatorname{div} \left(\frac{v}{\sqrt{1+|v|^2}} \right) = 0 < \frac{1}{\sqrt{1+|u_n|^2}} = \operatorname{div} \left(\frac{u_n}{\sqrt{1+|u_n|^2}} \right)$$

and $\liminf(v - u_n) \geq 0$ on $\partial\Omega \setminus E$, where E is the set of vertices of Ω, Lemma 9 implies $v > u_n$ for all n. Hence $\lim_{n\to\infty} u_n = u$ is the desired solution.

□

Remark

Here it is not reasonable to expect a "full" Jenkins-Serrin result with boundary values $-\infty$. The reason for this is that $M \times \mathbb{R}$ is not complete when equipped with the Ilmanen's metric g_c (geodesics going down have finite length).

We also have the following structural result.

Theorem 11 (Gama-H-Lira-Martín)

Let M be a complete Riemannian manifold and $\Omega \subset M$ be a domain (not necessarily regular). Let $\Lambda \subset \partial\Omega$ be a smooth open set and Σ a translating or minimal graph of a smooth function $u \colon \Omega \to \mathbb{R}$ that is complete as we approach Λ. Then $H_\Lambda = 0$.

Idea of the proof is to fix $x_0 \in \Lambda$ and take a sequence $x_i \to x_0$ with $u(x_i) \to \infty$. Then, with compactness results from geometric measure theory, we get that

$$S_i = \mathrm{Graph}[u - u(x_i)] \to S_\infty$$

to a minimal surface (w.r.t. g_c) in a ball $B \subset M \times \mathbb{R}$ centered at $(x_0, 0)$.

To conclude we show that a neighbourhood of $(x_0, 0)$ in S_∞ lies on $\Lambda \times \mathbb{R}$, and since $\tilde{H}_{\Lambda\times\mathbb{R}}(x,t) = e^{-ct/m} H_\Lambda(x)$, it follows that $H_\Lambda = 0$. □

Horizontal translating graphs

Let M^2 be a 2-dimensional complete Riemannian surface and Z a non-singular Killing vector field in M^2. Then the lift $Z(p,t) := Z(p)$, $(p,t) \in M^2 \times \mathbb{R}$ is a Killing field in $M^2 \times \mathbb{R}$ endowed with the product metric $g_0 := \sigma + dt^2$.

Remark

Recal that Z is a Killing field if the flow generated by Z is an isometry.

Let $\mathbb{P}$ be a fixed totally geodesic leaf of the orthogonal distribution associated to Z in $M^2 \times \mathbb{R}$.

Since Z is a lifting of a Killing field in M^2, we have $\mathbb{P} = \Gamma \times \mathbb{R}$, where Γ is a geodesic in M.

We will denote by $\Psi \colon \mathbb{P} \times \mathbb{R} \to M \times \mathbb{R}$ the flow generated by Z.

This flow gives local coordinates: If x is a coordinate in Γ, we can describe a point $p \in M^2 \times \mathbb{R}$ using the flow of Z, i.e. $p = \Psi((x,t),s)$. Therefore (x,t,s) is a local coordinate for $\mathbb{P} \times \mathbb{R} = M^2 \times \mathbb{R}$.

The corresponding coordinate vector fields are

$$\begin{aligned}
\partial_s(x,t,s) &= Z(\Psi((x,t),s));\\
\partial_t(x,t,s) &= \Psi_*((x,t),s)\partial_t(x,t);\\
\partial_x(x,t,s) &= \Psi_*((x,t),s)\partial_x(x,t),
\end{aligned}$$

and the components of the product metric are given by

$$\begin{aligned}
g_{11} &= \langle \partial_s, \partial_s \rangle =: \rho^2(x), \quad g_{12} = \langle \partial_s, \partial_x \rangle = 0, \quad g_{13} = \langle \partial_s, \partial_t \rangle = 0\\
g_{22} &= \langle \partial_x, \partial_x \rangle = \varphi^2(x), \quad g_{23} = \langle \partial_t, \partial_x \rangle = 0, \quad g_{33} = \langle \partial_t, \partial_t \rangle = 1.
\end{aligned}$$

Therefore

$$g_0 = \varphi^2(x)dx^2 + \rho^2(x)ds^2 + dt^2,$$

i.e. $M^2 \times \mathbb{R}$ is locally a warped product, and from now on we consider $M^2 = S \times_\rho \mathbb{R}$, where S may be either $\mathbb{S}^1$ or $\mathbb{R}$ endowed with a Riemannian metric $\varphi^2(x)dx^2$ and ρ is a positive smooth function in S.

With this convention $\mathbb{P} = S \times \mathbb{R}$, with Riemannian metric $h_0 := \varphi^2(x)dx^2 + dt^2$ and $M^2 \times \mathbb{R} = \mathbb{P} \times_\rho \mathbb{R}$.

A horizontal graph "over" a domain $\Omega \subset \mathbb{P}$ means the surface $\Sigma \subset M^2 \times \mathbb{R}$ (Killing graph) given by

$$\Sigma = \{\Psi(x, t, u(x, t)) \in \mathbb{P} \times_\rho \mathbb{R} \colon (x, t) \in \Omega\},$$

where $u \colon \Omega \to \mathbb{R}$ is a smooth function.

The conformal change to the Ilmanen's metric can be now written as

$$g_c = e^{ct}(\varphi^2(x)dx^2 + dt^2 + \rho^2(x)ds^2) =: h_c + e^{ct}\rho^2(x)ds^2,$$

where h_c denotes the restriction of Ilmanen's metric g_c to $\mathbb{P}$ (note that g_c is still a warped metric).

From now on we will always consider the metric h_c in $\mathbb{P}$ and the metric g_c in $M \times \mathbb{R}$. Also, to simplify the notation we will denote by $f\colon M \times \mathbb{R} \to \mathbb{R}$ the function

$$f(x,t) = e^{\frac{c}{2}t}\rho(x).$$

Remark

Ilmanen's metric is not complete in $M \times \mathbb{R}$ but we will need that $(M \times \mathbb{R}, g_0)$ is complete.

Let $\Sigma \subset M \times \mathbb{R}$ be a horizontal translating graph ($\mathbf{H} = cX^{\perp}$) of a function $u\colon \Omega \subset \mathbb{P} \to \mathbb{R}$. Then Σ can be oriented by the unit normal vector field

$$N = \frac{1}{f}\frac{\partial_s}{W} - f\frac{\nabla u}{W},$$

where we denote by ∇u the translation $\Psi_* \nabla u$ from $x \in \Omega$ to the point $\Psi(x, u(x)) \in \Sigma$.

From (2) we see that in Ω u satisfies the PDE

$$\operatorname{div}_{\mathbb{P}} \left(f^2 \frac{\nabla u}{W} \right) = 0, \quad W = \sqrt{1 + f^2 h_c(\nabla u, \nabla u)}, \tag{5}$$

where the gradient and divergence are taken w.r.t. the metric h_c in $\mathbb{P}$.

Another (maybe more familiar to people working with Killing graphs) way to write (5) is

$$
\begin{aligned}
0 &= \operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) = \frac{1}{f}\operatorname{div}_{\mathbb{P}}\left(f^2\frac{\nabla u}{W}\right) \\
&= \frac{1}{f}\operatorname{div}_{\mathbb{P}}\left(f\frac{\nabla u}{\sqrt{f^{-2}+h_c(\nabla u,\nabla u)}}\right) \\
&= \operatorname{div}_{\mathbb{P}}\left(\frac{\nabla u}{\sqrt{f^{-2}+h_c(\nabla u,\nabla u)}}\right) + \left\langle \nabla \log f, \frac{\nabla u}{\sqrt{f^{-2}+h_c(\nabla u,\nabla u)}}\right\rangle \\
&= \operatorname{div}_{\mathbb{P},-\log f}\left(\frac{\nabla u}{\sqrt{f^{-2}+h_c(\nabla u,\nabla u)}}\right),
\end{aligned}
$$

where $\operatorname{div}_{\mathbb{P},-\log f}$ denotes the weighted divergence operator.

Recall the notation: $f(x,t) = e^{\frac{c}{2}t}\rho(x)$, ∇ denotes the connection in $(\mathbb{P}, h_c = g_c|\mathbb{P})$, and $\bar{\nabla}$ the connection in $(M \times \mathbb{R}, g_c)$.

We will be working with a special type of curves that requires the following definitions.

Definition 12

Let $\gamma\colon [0,1] \to M \times \mathbb{R}$ be a parametrized curve in $M \times \mathbb{R}$. Then the f-length of γ is

$$\mathrm{L}_f[\gamma] = \int_0^1 f(\gamma(r)) \sqrt{g_c(\gamma'(r), \gamma'(r))_{\gamma(r)}}\, dr.$$

Definition 13

Let γ be a curve in $M \times \mathbb{R}$. We say that γ is an f-geodesic if

$$\bar{\nabla}_r \gamma' = g_c(\gamma', \gamma') \frac{\bar{\nabla} f}{f} - 2 g_c \left(\frac{\bar{\nabla} f}{f}, \gamma' \right) \gamma', \tag{6}$$

where $\bar{\nabla}_r \gamma'$ denotes the covariant derivative of γ' along γ w.r.t. g_c.

Definition 14 (f-curvature)

Let γ be a curve in $\mathbb{P}$. The (scalar) f-curvature of γ is

$$\mathrm{k}_f[\gamma] := \mathrm{k}_{h_c}[\gamma] - h_c \left(\frac{\nabla f}{f}, N \right), \tag{7}$$

where $\mathrm{k}_{h_c}[\gamma]$ is the geodesic curvature of γ in $(\mathbb{P}, h_c)$ and $N \in T\mathbb{P}$ the unit normal along γ.

Some remarks

1. Let γ be a curve in $\mathbb{P}$. Consider the surface $\gamma \times \mathbb{R} = \Psi(\gamma, \mathbb{R})$ ruled by the flow lines of ∂_s passing through γ. Then

$$\mathrm{k}_f[\gamma] = \mathrm{H}_{\gamma \times \mathbb{R}},$$

where $\mathrm{H}_{\gamma \times \mathbb{R}}$ is the mean curvature of $\gamma \times \mathbb{R}$ in $(M \times \mathbb{R}, g_c)$.

Therefore there is a correspondence between f-geodesics and minimal cylinders in $M \times \mathbb{R}$.

2. By the definition a curve γ in $\mathbb{P}$ is an f-geodesic in $\mathbb{P}$ only if γ is an f-geodesic in $M \times \mathbb{R}$.

Some remarks

3. Let γ be a curve in $\mathbb{P}$ and consider the Killing rectangle over γ, with height h, defined by

$$\gamma \times [0,h] := \Psi(\gamma,[0,h]) = \{\Psi(p,s) \in \mathbb{P} \times_\rho \mathbb{R} \colon p \in \gamma,\, s \in [0,h]\}.$$

Then we have

$$\mathrm{Area}[\gamma \times [0,h]] = \int_0^1 \int_0^h f(\gamma(r)) \sqrt{h_c(\gamma'(r),\gamma'(r))}\, dr dz = h \mathrm{L}_f[\gamma].$$

Note that the length of a segment $\{\Psi((x,t),s) \colon s \in [0,h]\}$ of a flow line through the point $(x,t) \in \mathbb{P}$ is given by $hf(x,t)$.

Remark

The existence of (at least short) f-geodesics follows from the general theory of Riemannian manifolds: Let

$$\sigma_c := f^2 g_c = e^{2\log f} g_c$$

and denote by $\tilde{\nabla}$ the Riemannian connection in $M \times \mathbb{R}$ with the metric σ_c.

Since, under the conformal change, the connection changes by

$$\tilde{\nabla}_Y X = \bar{\nabla}_Y X + g_c\left(X, \frac{\bar{\nabla} f}{f}\right) Y + g_c\left(Y, \frac{\bar{\nabla} f}{f}\right) X - g_c\left(X, Y\right) \frac{\bar{\nabla} f}{f}$$

we conclude from (6) *that f-geodesics are geodesics in $(M \times \mathbb{R}, \sigma_c)$.*

Definition 15 (Admissible domain)

Let $\Omega \subset \mathbb{P}$ be a precompact domain. We say that Ω is an admissible domain if $\partial\Omega$ is a union of f-geodesic arcs $A_1, \ldots, A_s$, $B_1 \ldots, B_r$, f-convex arcs $C_1, \ldots, C_t$, and the end points of these arcs and no two arcs A_i and no two arcs B_i have a common endpoint.

Definition 16 (Admissible polygon)

Let Ω be an admissible domain. Then $\mathcal{P}$ is an admissible polygon if $\mathcal{P} \subset \overline{\Omega}$, the boundary of $\mathcal{P}$ is formed by edges of $\partial\Omega$ and f-geodesic segments, and the vertices of $\mathcal{P}$ are chosen among the vertices of Ω.

Let $\Omega \subset \mathbb{P}$ be an admissible domain with $\partial\Omega = \cup_i J_i$ s.t. $\{J_i\} \subset \partial\Omega$ satisfies

$$J_i \cap J_{i+1} = \alpha_i, \;\; i \in \{1, v-1\}, \quad \text{and} \quad J_v \cap J_1 = \alpha_v,$$

where α_i denotes the end point of J_i.

Let $c = \{c_i \colon J_i \to \mathbb{R}\}$ be a family of bounded continuous functions and $\gamma_c \subset \partial\Omega \times \mathbb{R} = \Psi(\partial\Omega, \mathbb{R})$ given by $\gamma_c(x) = \Psi(x, c_i(x))$ if $x \in \operatorname{int} J_i$ and γ_c is a horizontal line from $\Psi(\alpha_i, c_i(\alpha_i))$ to $\Psi(\alpha_i, c_{i+1}(\alpha_i))$ for $x = \alpha_i$.

Theorem 17

Let $\Omega \subset \mathbb{P}$ be a geodesically f-convex and an admissible domain as above. Let $c = \{c_i \colon J_i \to \mathbb{R}\}$ be a family of bounded continuous functions and γ_c the curve associated to c. Then there exists a unique solution of (5) *with boundary data γ_c.*

Now, let $\Omega \subset \mathbb{P}$ be an admissible domain so that

$$\partial\Omega = \left(\bigcup_{i=1}^{l} A_i\right) \bigcup \left(\bigcup_{j=1}^{t} B_j\right) \bigcup \left(\bigcup_{k=1}^{z} C_k\right),$$

where A_i and B_j are f-geodesic arcs and C_k are f-convex arcs. Let $\mathcal{P}$ be an admissible polygon. Then we denote

$$\alpha_f(\mathcal{P}) = \sum_{A_i \subset \partial\mathcal{P}} \mathrm{L}_f[A_i] \quad \text{and} \quad \beta_f(\mathcal{P}) = \sum_{B_i \subset \partial\mathcal{P}} \mathrm{L}_f[B_i].$$

Recall that the Jenkins-Serrin conditions for a solution $u\colon \Omega \to \mathbb{R}$ are

$$u|_{C_k} = c_k \quad u|_{A_i} = +\infty, \quad \text{and} \quad u|_{B_j} = -\infty.$$

If $\{C_k\} = \varnothing$, then we only require that $u \to +\infty$ on A_i and $u \to -\infty$ on B_j.

Theorem 18 (Gama-H-Lira-Martín)

Let $\Omega \subset \mathbb{P}$ be an admissible domain such that for any admissible polygon $\mathcal{P} \subset \bar{\Omega}$ we have

$$2\alpha_f(\mathcal{P}) < \mathrm{L}_f[\partial\mathcal{P}] \quad \textit{and} \quad 2\beta_f(\mathcal{P}) < \mathrm{L}_f[\partial\mathcal{P}]. \tag{8}$$

Then

(a) *If $\{C_k\} \neq \varnothing$ and $c_k \colon C_k \to \mathbb{R}$ are continuous functions, then there exists a Jenkins-Serrin solution of (5) with continuous boundary data c_k.*

(b) *If $\{C_k\} = \varnothing$ and $\alpha_f(\partial\Omega) = \beta_f(\partial\Omega)$, then there exists a Jenkins-Serrin solution of (5).*

Furthermore, if u is a Jenkins-Serrin solution of (5) with continuous boundary data

$$c_k \colon C_k \to \mathbb{R}$$

and if $\{C_k\} \neq \varnothing$, then inequalities (8) hold for all admissible polygon $\mathcal{P}$, and if $\{C_k\} = \varnothing$ then we also have $\alpha_f(\partial\Omega) = \beta_f(\partial\Omega)$.

Brief idea of the proof. Theorem 17 gives the existence of solutions (with finite boundary data) only over f-convex domains, so we use Perron's method to obtain the existence over more general admissible domains.

If $\Omega \subset \mathbb{P}$ is a domain with C^1 smooth boundary $\partial\Omega$ and u solution of (5), we have

$$\int_{\partial\Omega} \frac{f^2}{W} h_c(\nabla u, \nu) = 0.$$

This motivates to define the flux formula

$$F_u[\gamma] = \int_{\gamma} \frac{f^2}{W} h_c(\nabla u, \nu),$$

which plays important role in the study of the divergence set $\mathcal{D}$ and in the proof of the theorem.

Using the flux (among other things) we can prove that any connected component of the divergence set $\mathcal{D}$ is an admissible polygon in Ω.

This property, together with the flux, imply that if the structural condition (8) is satisfied, then $\mathcal{D} = \emptyset$ and we obtain a solution.

On the other hand, the flux formula can be used to show that the existence of a solution implies the structural condition (8).

Examples in $\mathbb{R}^3$

In this case $\mathbb{P}$ is a vertical plane ($\mathbb{R}^2$) containing the vector e_3 in $\mathbb{R}^3$ and the Ilmanen's metric is given by $g_c = e^{cx_3}\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$.

Therefore the function f is given by $f = e^{c\frac{x_3}{2}}$, and γ is an f-geodesic in $\mathbb{P}$ if and only if γ satisfies

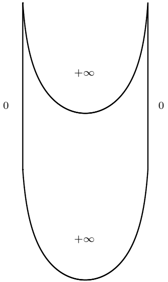
$$\mathrm{k}[\gamma] = c\langle N, e_3\rangle,$$

where $\mathrm{k}[\gamma]$ denotes the scalar curvature of γ in $\mathbb{P}$, N denotes the unit normal to γ and $\langle\cdot,\cdot\rangle$ is the Euclidean metric of $\mathbb{P} = \mathbb{R}^2$.

In particular, f-geodesics are translating curves in $\mathbb{R}^2$.

Assume now that $c = 1$. It is well known that the unique translating curves are vertical lines in the direction e_3 and the grim reaper curves

$$x_3 = -\log\cos x_1, \quad x_1 \in (-\pi/2, \pi/2).$$



Therefore we can produce admissible domains $\Omega \subset \mathbb{P}$ that are bounded by vertical line segments and parts of the grim reaper curves.

If we assign boundary data $+\infty$ on the parts of the grim reaper curve (edges A_1, A_2) and continuous data on the vertical segments (edges C_1, C_2), the condition for the existence of solutions becomes

$$\mathrm{L}_f[A_1] + \mathrm{L}_f[A_2] < \mathrm{L}_f[C_1] + \mathrm{L}_f[C_2].$$

Example 19

For the edges of $\Omega \subset \mathbb{P}$, we can take the parametrizations

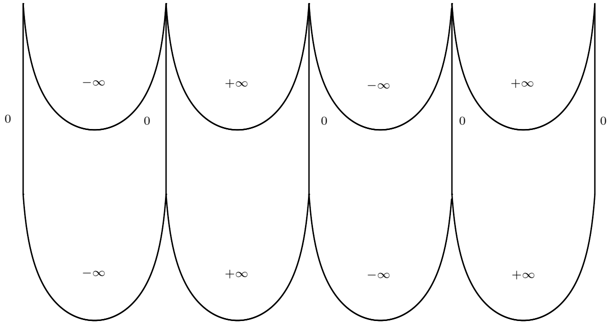
$$
\begin{aligned}
A_1 &= \{(x_1, 0, a - \log\cos x_1) \colon x_1 \in (r, s)\};\\
A_2 &= \{(x_1, 0, b - \log\cos x_1) \colon x_1 \in (r, s)\};\\
C_1 &= \{(r, 0, x_3) \colon x_3 \in (a - \log\cos s, b - \log\cos s)\};\\
C_2 &= \{(s, 0, x_3) \colon x_3 \in (a - \log\cos r, b - \log\cos r)\},
\end{aligned}
$$

$-\pi/2 < s < r < \pi/2$, $a, b \in \mathbb{R}$ and $a < b$.
Then

$$
\begin{aligned}
\mathrm{L}_f[A_1] + \mathrm{L}_f[A_2] &= (e^b + e^a)(\tan r - \tan s)\\
\mathrm{L}_f[C_1] + \mathrm{L}_f[C_2] &= (e^b - e^a)(\sec r + \sec s).
\end{aligned}
$$

Fixing $a < b$ and choosing $r - s > 0$ small enough, we have $\mathrm{L}_f[A_1] + \mathrm{L}_f[A_2] < \mathrm{L}_f[C_1] + \mathrm{L}_f[C_2]$. And changing C_i to B_i we can get $\mathrm{L}_f[A_1] + \mathrm{L}_f[A_2] = \mathrm{L}_f[B_1] + \mathrm{L}_f[B_2]$ for the case (b) in Theorem 18.

Note that the reflection with respect to the plane $\mathbb{P}$ is an isometry and therefore by reflecting the previous "basic solution", we can obtain a periodic surface with alternating boundary values $+\infty$ and $-\infty$.



Thank you!