Minimal surfaces in $H^2 \times R$
with finite total curvature
and related problems

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Theorem (Hauswirth-Rosenberg, 2006)

\( \Sigma \subset \mathbb{H}^2 \times \mathbb{R} \) compl. or. min. surf.

\( |\int_{\Sigma} K| < +\infty, \quad K = \text{Gauss curv. of} \ \Sigma \)

- \( \Sigma^{conf} \cong \mathbb{M} - \{p_1, \ldots, p_k\} \).
- \( Q = \text{Hopf diff. of} \ \Sigma \to \mathbb{H}^2 \) extends meromorphically to \( \mathbb{M} \);
  \( Q(z) = z^{2m_i}(dz)^2 \) at \( p_i, \ m_i \geq 0 \).
- \( N_3 \to 0 \) at \( p_i \).
- \( \int_{\Sigma} K = 2\pi(2 - 2g - 2k - \sum_{i=1}^{k} m_i) \).
**Theorem (Hauswirth-Rosenberg, 2006)**

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- \( \int_{\Sigma} K = 2\pi(2 - 2g - 2k - \sum_{i=1}^{k} m_i). \)
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Examples: Scherk graphs over ideal polygons with $2k$ edges, $k \geq 2$ (J-S condition) $\sim \int_{\Sigma} K = 2\pi(1 - k)$
Question [Hauswirth-Rosenberg]:

Are there non-symply connected examples of f.t.c.?

An annulus $\Sigma$ with $\int_\Sigma K = -4\pi$?
New examples

$k$-noids

Theorem (Pyo, Morabito - _ __)

For any $k \geq 2$, $\exists \Sigma_k \subset \mathbb{H}^2 \times \mathbb{R}$ PEMS with genus 0, $k$ vertical planar ends and

$$\int_{\Sigma} K = 4\pi(1 - k).$$

($\exists$ a $(2k - 3)$-parameter family)
New examples

$k$-noids

Parameter = dist. between the asymptotic vertical planes
New examples

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We can take limits of $k$-noids, $k \to +\infty$

Question [Ros]: Is there a PEMS for any genus 0 topology?

Theorem (Martín - __)

\[ \forall \Sigma = \text{planar domain}, \exists f : \Sigma \to \mathbb{H}^2 \times \mathbb{R} \text{ prop. min. embedding.} \]

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Question: Are the geodesics defining the ends "ordered"?
Question: Examples with higher genus?
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Theorem (Martín - Mazzeo - __)

For any \( g \geq 0 \) and \( k > 1 \) large, \( \exists \Sigma_{g,k} \subset \mathbb{H}^2 \times \mathbb{R} \ PEMS \) with f.t.c., genus \( g \) and \( k \) vertical planar ends.

Moreover, the c.c. of

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\mathcal{M}_{g,k} = \left\{ \Sigma \subset \mathbb{H}^2 \times \mathbb{R} \ PEMS \text{ with f.t.c., genus } g \text{ and } k \text{ vertical planar ends} \right\}
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containing \( \Sigma_{g,k} \) is a real analytic space of dimension \( 2k - 3 \).
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Minimal surfaces in $H^2 \times R$ with f.t.c.
Classification results

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Theorem (Hauswirth - Sa Earp - Toubiana)
\[ \Sigma = \text{min. surf. in } \mathbb{H}^2 \times \mathbb{R} \text{ with } \int_{\Sigma} K = 0 \Rightarrow \Sigma = \text{vert. plane} \]

Theorem (Pyo - )
\[ \Sigma = \text{min. surf. in } \mathbb{H}^2 \times \mathbb{R} \text{ with f.t.c. } \int_{\Sigma} K = -2\pi \]
\[ \Rightarrow \Sigma = \text{a Scherk minimal graph over an ideal quadrilateral} \]

Theorem (Hauswirth - Nelli- Sa Earp - Toubiana)
\[ \Sigma = \text{min. surf. in } \mathbb{H}^2 \times \mathbb{R} \text{ with f.t.c. and 2 vertical planar ends} \]
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**Theorem (Colding-Minicozzi)**

Any compl. emb. min. surf. with fin. top. in $\mathbb{R}^3$ must be proper.

**Generalizations**

$\rightsquigarrow$ Meeks-Rosenberg, Meeks-Pérez-Ros

**Theorem (Coskunuzer)**

There exists a compl. non-proper emb. min. disk in $\mathbb{H}^3$.

**Theorem (___ - Tinaglia)**

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Calabi-Yau problem

Minimal surfaces in $H^2 \times \mathbb{R}$ with f.t.c.
Saddle Towers and minimal k-noids in $\mathbb{H}^2 \times \mathbb{R}$
(joint work with Filippo Morabito),

Minimal surfaces with limit ends in $\mathbb{H}^2 \times \mathbb{R}$,

Non-simply connected minimal planar domains in $\mathbb{H}^2 \times \mathbb{R}$
(joint work with Francisco Martín), to appear in Trans. AMS.

Minimal surfaces with positive genus and finite total curvature
in $\mathbb{H}^2 \times \mathbb{R}$ (joint work with Francisco Martín and
Rafe Mazzeo), preprint.

Simply-connected minimal surfaces with finite total curvature
in $\mathbb{H}^2 \times \mathbb{R}$ (joint work with Juncheol Pyo),

Non-proper complete minimal surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$
(joint work with Giuseppe Tinaglia), preprint.