Sobre la clasificación de los espacios lorentzianos r-ésimo simétricos

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Aim of the talk:

- To classify the 2nd-symmetric Lorentzian manifolds, i.e.:

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  \[ \nabla^2 R := \nabla(\nabla R) = 0 \]

- To provide properties and open questions on the \emph{rth-symmetric} case \( \nabla^r R = 0 \) and, in general on the implications of \( \nabla^r T = 0 \) for any tensor field.
Introduction

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- Penrose limit type constructions
- “Super-energy” tensor
- Higher order Lagrangian theories, supergravity, vanishing of quantum fluctuations...
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- So, instead of $\nabla^2 R = 0$, **semi-symmetric spaces** were introduced (Cartan, Szabó):

\[
\begin{align*}
\nabla^2 R(X, Y; \ldots) - \nabla^2 R(Y, X; \ldots) &= \\
&= \nabla_X (\nabla_Y R) - \nabla_Y (\nabla_X R) - \nabla_{[X,Y]} R \\
&=: R(X, Y) \cdot R = 0
\end{align*}
\]
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\]

- **Lorentzian and higher signatures**: $\nabla^r R = 0 \not\Rightarrow \nabla R = 0$
Introduction

- So, a ladder of conditions appear in the Lorentzian case:
  - Locally symmetric $\subset$ 2nd-symmetric $\subset$ semi-symmetric
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How hadn’t 2nd-symmetry been studied before?
Introduction

Main result to be proven:

**Theorem (Blanco, Senovilla, —)**

Let \((M, g)\) be a *properly 2nd-symmetric* Lorentzian \(n\)-manifold:
- (Local classification). \((M, g)\) is *locally isometric to a product*
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*Let* $(M, g)$ *be a properly 2nd-symmetric* Lorentzian $n$-*manifold:*

- (Local classification). $(M, g)$ is *locally isometric to a product*
  - a (non-flat) *symmetric Riemannian space* $(N, g_N)$
  - a *proper 2nd-order Cahen-Wallach space* $(\mathbb{R}^{d+2}, g_A)$,
  
    $g_A = -2 du \left( dv + (a_{ij} u + b_{ij}) x^i x^j du \right) + \delta_{ij} dx^i dx^j$

  *with some* $a_{ij} \neq 0$. 
Introduction

Main result to be proven:

**Theorem (Blanco, Senovilla, — )**

Let \((M, g)\) be a properly 2nd-symmetric Lorentzian \(n\)-manifold:

- **(Local classification).** \((M, g)\) is locally isometric to a product
  - a (non-flat) symmetric Riemannian space \((N, g_N)\)
  - a proper 2nd-order Cahen-Wallach space \((\mathbb{R}^{d+2}, g_A)\),
    \[ g_A = -2 du \left( dv + (a_{ij}u + b_{ij})x^i x^j du \right) + \delta_{ij} dx^i dx^j \]
    with some \(a_{ij} \neq 0\).

- **(Global classification).** Moreover, if \((M, g)\) is 1-connected and geodesically complete, then it is globally isometric to \((\mathbb{R}^{d+2} \times N, g_A \oplus g_N)\).
Characterizations of local symmetry vs 2nd-symmetry

Proposition
For a (connected) semi-Riemannian manifold \((\mathbb{N}, h)\), they are equivalent:

(i) \((\mathbb{N}, h)\) is locally symmetric, i.e. \(\nabla R = 0\).
(ii) If \(X, Y\) and \(Z\) are parallel vector fields along a curve \(\gamma\), then so is \(R(X, Y)Z\).
(iii) The sectional curvature of non-degenerate planes is invariant under parallel transport.
(iv) The local geodesic symmetry \(s_p\) is an isometry at any \(p \in \mathbb{N}\).
(v) \((\mathbb{N}, h)\) is locally isometric to a symmetric space.

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Lorentzian \(r\)-th symmetric spaces
Characterizations of local symmetry vs 2nd-symmetry

Local symmetry

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(v) $(N, h)$ is locally isometric to a symmetric space.
Characterizations of local symmetry vs 2nd-symmetry

Remark

“\((N, h)\) is locally isometric to a symmetric space”

\(\sim\) as a difference with the locally homogeneous case, as there exists even Riemannian non-regular ones (Kowalski’97)
Characterizations of local symmetry vs 2nd-symmetry

2nd symmetry

Lemma

For a semi-Riemannian \((N, h)\), they are equivalent:

- Skew symmetry of \(\nabla^2 R\) in the derivatives slots.

- For any non-degenerate tangent plane \(\Pi_p \subset T_pN\), its parallel transport \(\Pi_\gamma\) along any geodesic \(\gamma\), the derivative of its sectional curvature \(\frac{d}{d\tau}(K(\Pi_\gamma))\) is a constant along \(\gamma\).

- For any parallely propagated vector fields \(X, Y, Z\) along any geodesic \(\gamma\), the vector field \((\nabla_\gamma' R)(X, Y)Z\) is itself parallely propagated along \(\gamma\).
Characterizations of local symmetry vs 2nd-symmetry

**Proposition**

For a semi-Riemannian \((N, h)\), they are equivalent:

(i) \((N, h)\) is 2nd-symmetric, \(\nabla \nabla R = 0\)

(ii) \((N, h)\) is semi-symmetric \((R(X, Y)R = 0)\) and satisfies any of the equivalent conditions to skew-symmetry in the lemma.

(iii) If \(V, X, Y, Z\) are parallelly propagated vector fields along any curve, then so is \((\nabla_v R)(X, Y)Z\).
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Remark

Characterizations in terms of an analog of the geodesic symmetry or local isometries to a model space are conspicuously absent.
Classification locally symmetric vs 2nd-symmetric

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**Proof**: Use de Rham decomposition
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Proof. 1. Ricci is parallel, so use classical Eisenhart theorem:

- If a Riemannian \((N, g_R)\) admits a 2-cov. symmetric parallel \(L\).
  - \(L \neq cg_R\), then locally:
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     - \(g_R\) is reducible: \(g_R = g_R^{(1)} \oplus g_R^{(2)} \oplus \ldots \oplus g_R^{(s)}\).
     - \(L = \sum_{m=1}^{s} \lambda_m g_R^{(m)}\) for some \(\lambda_m \in \mathbb{R}\).
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2. Holds even for homogeneous sp. (Alekseevsky, Kimelfeld ’75)
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2. Holds even for homogeneous sp. (Alekseevsky, Kimelfeld ’75)—and locally homogeneous with \(\text{Ric} \leq 0\) are regular (Spiro ’93)
Classification locally symmetric vs 2nd-symmetric

Lorentzian symmetric spaces

**Theorem (Cahen, Wallach '70)**

A *complete 1-connected Lorentzian symmetric space* \((M, g)\) is isometric to the *product* of a *simply-connected Riemannian symmetric space* and one of the following Lorentzian manifolds:
Classification locally symmetric vs 2nd-symmetric

**Lorentzian symmetric spaces**

**Theorem (Cahen, Wallach '70)**

A complete 1-connected Lorentzian symmetric space \((M, g)\) is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:

1. \((\mathbb{R}, -dt^2)\)
2. The universal cover of de Sitter or anti-de Sitter \(d\)-spaces, \(d \geq 2\),
3. A Cahen-Wallach space \(CW^d(A) = (\mathbb{R}^d, g_A), d \geq 2\), where \(A \equiv (A_{ij})\) is a \((d - 2) \times (d - 2)\) matrix and

\[
g_A = -2du \left( dv + A_{ij}x^ix^jdu \right) + \sum_{ij} \delta_{ij}dx^idx^j
\]
Local symmetry vs. 2nd-symmetry
When $\nabla' T = 0 \Rightarrow \nabla T = 0$?
Brinkmann spaces
Sketch of proof

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Remark
Choosing $A$ with $\text{trace}(A) = 0$:
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there are Ricci flat non-flat Lorentzian symmetric spaces.

Remark

Lorentzian symmetric space with a parallel lightlike v.f. $K$ ⇒:
Locally isometric to the product of a $CW^d(A), d > 2$ and
Riemannian symmetric space.
Classification locally symmetric vs 2nd-symmetric

2nd-symmetric:
The theorem to be proven shows:

**proper 2nd-symmetric spaces only appear generalizing the family of Cahen-Wallach spaces** \( CW^d(A), d > 2 \):

- \( \sim \rightarrow \) allow an affine dependence of the matrix \( A \) in \( u \)
Generalization of Cahen-Wallach family

Generalized Cahen-Wallach $d$-space of order $r$, $\text{CW}_r^d(A) = (\mathbb{R}^d, g_A)$, $d \geq 2$: metric:

$$g_A = -2du \left( dv + \sum_{ij} A_{ij}(u)x^i x^j du \right) + \sum_{ij} \delta_{ij} dx^i dx^j$$

where $A \equiv (A_{ij}(u))$ is a $(d - 2) \times (d - 2)$ matrix:

$$A_{ij}(u) = A_{ij}^{(r-1)} u^{r-1} + \cdots + A_{ij}^{(1)} u + A_{ij}^0$$

for symmetric (constant) matrixes $A_{ij}^k$. 
Proposition

Any generalized Cahen-Wallach space $\text{CW}_r^d(A)$ satisfies:

1. If $A_{ij}^{(r-1)} \neq 0$ (\(\text{CW}_r^d(A)\) is proper) then it is proper $r$th-symmetric

1. Direct computation: in an appropriate basis

\[ \{E_\alpha\} = \{E_0 = \partial_u - \sum A_{ij} x^i x^j \partial_v, E_1 = \partial_v, \partial_i\} \]

the only non-vanishing components of $\nabla^l R, l \in \{0, \ldots, r-1\}$ are:

\[ \nabla_0^{(l)} \cdot \nabla_0 R^{1}_{i0j} = \frac{d^l A_{ij}}{d u} = \sum_{k=l}^{r-1} \frac{k!}{(k-l)!} A_{ij}^{(k)} u^{k-l} \]
Generalization of Cahen-Wallach family

Proposition

Any generalized Cahen-Wallach space $CW_{r}^{d}(A)$ satisfies:

1. If $A_{ij}^{(r-1)} \neq 0$ ($CW_{r}^{d}(A)$ is proper) then it is proper $r$th-symmetric
2. $K = \partial_{\nu}$ is a lightlike parallel vector field
3. It is analytic
4. It is geodesically complete

Proof. 2,3: Trivial
Local symmetry vs. 2nd-symmetry
When $\nabla^r T = 0 \Rightarrow \nabla T = 0$?
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Generalization of Cahen-Wallach family

Proposition

Any generalized Cahen-Wallach space $CW^d_r(A)$ satisfies:

1. If $A^{(r-1)}_{ij} \neq 0$ ($CW^d_r(A)$ is proper) then it is proper $r$th-symmetric
2. $K = \partial_v$ is a lightlike parallel vector field
3. It is analytic
4. It is geodesically complete

Proof. 2,3: Trivial
4. Direct computation or general results (Candela, Romero, — ’13)
Corollary

A complete 1-connected Lorentzian manifold locally isometric to some $CW_r^d(A)$ is globally isometric too.

This will allow to go from the local to the global result.
Must rth-symmetry imply local symmetry?

This is a particular case of:

- When $\nabla^r T = 0 \Rightarrow \nabla T = 0$?
Riemannian case

**Theorem**

Let \((M, g)\) be *Riemannian* and \(T\) a tensor field such that \(\nabla'r T = 0\). Then \(\nabla T = 0\) if either

(a) (Nomizu-Ozeki '62) \(g\) is complete and irreducible, or

(b) (Nomizu [unpub], Tanno '72) \(T\) is \(R\), or \(Ric\), Weyl, projective \(t\).
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**Remark**

In particular, from (b), Riemannian \(r\)-th symmetric implies locally symmetric.
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Proof (a) 1. Case \(r = 2\) suffices (replace otherwise \(\widetilde{T} := \nabla^{r-2} T\)).
2. Put \(f := g(T, T)/2\). Using \(\nabla^2 T = 0\):

\[
\text{Hess} f(X, Y) = g(\nabla_X T, \nabla_Y T) \quad \text{and} \quad \nabla \text{Hess} f = 0
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3. By Eisenhart thm: \(\text{Hess} f = cg\), \(c \in \mathbb{R}\). Thus \(Z := \text{grad}(f)\) satisfies \(\nabla_X Z = cX\) (in particular, is homothetic)
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4. Under irreducibility + completeness homothetic vectors are Killing: \(c = 0\) \(g(\nabla_X T, \nabla_Y T) = 0\). As \(g\) is Riemannian \(\nabla T = 0\).
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Let \((M, g)\) be **Riemannian** and \(T\) a tensor field such that \(\nabla' T = 0\). Then \(\nabla T = 0\) if either

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**Proof (b)** 1. Irreducibility can be assumed: \(T = 0\) on the flat part of (local) de Rham decomposition (as well as on mixed elements)
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2. As before, one has \(\nabla_X Z = cX\) and needs \(c = 0\).
3. As \(Z\) is homothetic, it is affine. Thus \(L_Z \nabla = 0 = L_Z T\) and:

\[
0 = L_Z \nabla T = \nabla_Z (\nabla T) + (s + 1)c \nabla T = (s + 1)c \nabla T
\]

\((s: \text{covar minus contrav slots for } T)\). That is, if \(c \neq 0\) directly \(\nabla T = 0\). □
Conclusion

Remark

\[ \nabla' T = 0 \nRightarrow \nabla T = 0 \] only when:

- The manifold is reducible, with a flat part in de Rham decomposition, OR
- The manifold is incomplete with a proper (non-Killing) homothetic vector field (necessarily without zeroes)

In the latter case the metric can be written locally as a cone:

\[ M = I \times S, \quad I \subset (0, \infty), \] with

\[ g = dt^2 + t^2 \pi^* S \]

being Z = t \partial_t proper homothetic. In particular:

\[ \nabla Z = 2 \cdot \text{Id} (\neq 0) \]

\[ \nabla^2 Z = 0 \]
Remark

\[ \nabla' T = 0 \nleftrightarrow \nabla T = 0 \] only when:

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In the latter case the metric can be written locally as a *cone*:

$M = I \times S, I \subset (0, \infty), (S, g_S)$ Riemannian

$$g = dt^2 + t^2 \pi^*_S g_S$$

being $Z = t \partial_t$ proper homothetic. In particular:

$$\nabla Z = 2 \cdot \text{Id}(\neq 0) \quad \nabla^2 Z = 0$$
Difficulties for the semi-Riemannian extension

1. The (full, local) de Rham decomposition cannot be carried out when the subspaces invariant by local holonomy are degenerate.
2. The conclusion $c=0$ only means $g(T,T)$ constant and $g(\nabla T,\nabla T) = 0$ i.e. $\nabla T$ is a lightlike tensor.
3. Even in the non-degenerate irreducible case, to apply Eisenhart one needs: if the restricted homogeneous holonomy group is irreducible and a symm. 2-cov tensor $h$ is invariant by the group, then $h = cg$ for some function $c$, which is constant if $h$ is parallel.
   However, this holds in Lorentzian signature and others (Tanno'67, $n=2$ or non-neutral signature).

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1. The (full, local) de Rham decomposition cannot be carried out when the subspaces invariant by local holonomy are degenerate.

2. The conclusion $c = 0$ only means $g(T, T)$ constant and $g(\nabla T, \nabla T) = 0$ i.e. $\nabla T$ is a lightlike tensor.

3. Even in the non-degenerate irreducible case, to apply Eisenhart one needs: if the restricted homogeneous holonomy group is irreducible and a symm. 2-cov tensor $h$ is invariant by the group, then $h = cg$ for some function $c$, which is constant if $h$ is parallel. However, this holds in Lorentzian signature and others (Tanno’67, $n = 2$ or non-neutral signature).
Further properties: $\nabla^r T = 0$ in generic points

**Definition**

A point $p$ is generic if the curvature endomorphism:

$$R : \Lambda^2(M) \to \Lambda^2(M), \quad \nu^b \wedge w^b \mapsto 2R(\nu, w)$$

is an isomorphism when restricted to $p$.

**Theorem**

*If there exists a generic point, $\nabla^r T = 0$ implies $\nabla T = 0$, for any semi-Riemannian metric.*
\[ \nabla' T = 0 \text{ in generic points} \]

**Theorem**

*If there exists a generic point, \( \nabla' T = 0 \) implies \( \nabla T = 0 \), for any semi-Riemannian metric.*

**Proofs of increasing generality:**

1. Simply, *no conic metric* (nor flat one) *is generic.*
\n\n\n**Theorem**

*If there exists a generic point, \( \nabla' T = 0 \) implies \( \nabla T = 0 \), for any semi-Riemannian metric.*

**Proofs of increasing generality:**

1. Simply, **no conic metric** (nor flat one) is generic.

**Remarks**

- Valid **only for the Riemannian case**
- Extensible to generic (non-degenerate) Ric, as \( \text{Ric}(\partial_t, X) = 0 \) in the conic metric
Theorem

*If there exists a generic point, \( \nabla^r T = 0 \) implies \( \nabla T = 0 \), for any semi-Riemannian metric.*

Proofs of increasing generality:

1. (Tanno ’72) As we had \( Z \) with \( \nabla X Z = cX \):
   \[
   0 = L_Z \nabla = \nabla^2 Z + R(Z, \cdot) = R(Z, \cdot)
   \]
   So \( R \) is not invertible except if \( Z = 0 \).
Local symmetry vs. 2nd-symmetry

When $\nabla' T = 0 \Rightarrow \nabla T = 0$?

Brinkmann spaces

Sketch of proof

Riemannian case

Semi-Riemannian extension

Generic points

Old techniques and lightlike parallel vector fields

$\nabla' T = 0$ in generic points

Theorem

If there exists a generic point, $\nabla' T = 0$ implies $\nabla T = 0$, for any semi-Riemannian metric.

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So $R$ is not invertible except if $Z = 0$.

Remarks:

- Also valid for Riemannian and extensible to generic Ric
- For Lorentz and non-neutral sign. + irreducibility, it applies, but then implies only $g(\nabla T, \nabla T) = 0$ and $g(T, T) = \text{const.}$
\( \nabla^r T = 0 \) in generic points

**Theorem**

*(Senovilla '08)* If there exists a generic point, \( \nabla^r T = 0 \) implies \( \nabla T = 0 \) on all \( M \), for any semi-Riemannian metric.

Proofs of increasing generality:

3. *(Senovilla '08)* Apply the Ricci identities to \( T \) and \( \nabla T \): The invertibility of \( R \) allows to clear \( \nabla T = 0 \).
\[ \nabla^r T = 0 \text{ in generic points} \]

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Proofs of increasing generality:

1. (Senovilla '08) Apply the Ricci identities to \( T \) and \( \nabla T \):
   The invertibility of \( R \) allows to clear \( \nabla T = 0 \).

Remarks:

- Independent of both, signature or previous computations
- Extensible to: all semi-symmetric spaces have constant curvature around generic points
Limits of old techniques

A computation in the spirit of old papers:

Proposition

Let \((M, g)\) be semi-Riemannian and \(r\)-th symmetric. If there exists a vector field \(Z\):

\[
\nabla_X Z = cX \quad c \in \mathbb{R} \quad \forall X \in \mathfrak{X}(M)
\]

then either \(Z\) is parallel or \(R = 0\).
Limits of old techniques

A computation in the spirit of old papers:

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Let \((M, g)\) be semi-Riemannian and \(r\)-th symmetric. If there exists a vector field \(Z\):

\[
\nabla_X Z = cX \quad c \in \mathbb{R} \quad \forall X \in \mathfrak{X}(M)
\]

then either \(Z\) is parallel or \(R = 0\).

**Proof.** As \(Z\) is homothetic, \(L_Z \nabla = 0\), \(L_Z \nabla^k R^l_{ijk} = 0\) and:

\[
0 = L_Z (\nabla^{r-1} R) = \nabla_Z (\nabla^{r-1} R) + (1 + r)c \nabla^{r-1} R = (1 + r)c \nabla^{r-1} R
\]

So, if \(c \neq 0\), use induction. □
Limits of old techniques

Corollary

A proper $r$th-symmetric Lorentzian $(M, g)$ either admits a parallel lightlike direction or satisfies that $\nabla^{r-1} R$ is (parallel and) null and $g(\nabla^{r-2} R, \nabla^{r-2} R)$ is a constant.

Proof. The first possibility occurs either when degenerately reducible or when admits a lightlike parallel v.f.
Corollary

A proper $r$th-symmetric Lorentzian $(M, g)$ either admits a parallel lightlike direction or satisfies that $\nabla^{r-1}R$ is (parallel and) null and $g(\nabla^{r-2}R, \nabla^{r-2}R)$ is a constant.

Proof. The first possibility occurs either when degenerately reducible or when admits a lightlike parallel v.f. Otherwise, in each irreducible part, put again $T = \nabla^{r-2}R$, $f = g(T, T)$, $\text{Hess} f(X, Y) = g(\nabla_X T, \nabla_Y T)$ and $Z = \text{grad} f$. By previous Prop., necessarily $Z \equiv 0$. □
Limits of old techniques

Corollary

A proper rth-symmetric Lorentzian \((M, g)\) either admits a parallel lightlike direction or satisfies that \(\nabla^{r-1} R\) is (parallel and) null and \(g(\nabla^{r-2} R, \nabla^{r-2} R)\) is a constant.

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By previous Prop., necessarily \(Z \equiv 0. \square\)

Remark

Limit of “old” results: this suggests that at least 2nd-symmetric Lorentzian spaces must admit a parallel lightlike v.f. \(K\).
Existence of a lightlike parallel vector field

**Theorem**

*(Senovilla '08).* Any proper 2nd-symmetric Lorentzian space admits a unique lightlike parallel vector field $K$.

(Alternative proof by Aleksevski & Galaev, '11.)
Existence of a lightlike parallel vector field

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Steps of direct original proof (as simplified in Blanco’s thesis):
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Steps of direct original proof (as simplified in Blanco’s thesis):

- Previous result for $\exists$ parallel light. vector, not only a line:
  - $\exists$ Parallel $L \neq cg$ plus no decomposable (non-degenerately reducible) $\Rightarrow \exists!$ independent parallel lightlike vector $K$.

  (proof by discussing possible Segre types)
Existence of a lightlike parallel vector field

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- **Previous result for \( \exists \) parallel light. vector, not only a line:**
  - \( \exists \) Parallel \( L \neq cg \) plus no *decomposable* (non-degenerately reducible) \( \Rightarrow \exists! \) independent parallel lightlike vector \( K \).
  (proof by discussing possible Segre types)

**Uniqueness:** a linear combination of \( K_1 \pm K_2 \) would be (parallel and) timelike in contradiction with no-decompsability/properness.
Existence of a lightlike parallel vector field

Theorem

(Senovilla '08). Any proper 2nd-symmetric Lorentzian space admits a unique independent lightlike parallel vector field $K$.

- **Analyze the curvature concomitants** showing that, either such a $K$ exists, or they vanish:
  (a) 1-form concomitants of order $m$ and degree up to $m + 1$
  (b) scalar or 2-cov. concomitants of equal order and degree.
- **Using Ricci identity**, such restrictions force the existence of $K$
Brinkmann spaces

**Definition**

A **Brinkmann space** is any Lorentzian $n$-manifold endowed with a complete lightlike parallel vector field $K$. 
Brinkmann spaces

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**Brinkmann decomposition** $\{u, v\}$:

1. $K$ parallel: fix $u$ (up to a constant) s.t.: $K = \text{grad} u$
2. $K$ lightlike: $K \perp$ degenerate totally geodesic integrable foliation with leaves $\Sigma_u$
3. Choose a hypersurface $\Omega$ transverse to $K$ so that $\tilde{M} := \Sigma_{u=0} \cap \Omega$ is spacelike and transverse
4. Let $\phi$ the flow of $K$, define $v$ so that $\phi_{-v(p)}(p) \in \Omega$
Construction of the Brinkmann decomposition
Construction of a Brinkmann chart

- **Brinkmann chart** \( \{u, v, x^i\} \): complete \( u, v \) to a chart by choosing \( n - 2 \) coordinates \( x^i \) independent of \( u \) in \( \Omega \).
Construction of a Brinkmann chart

- **Brinkmann chart** \(\{u, v, x^i\}\): complete \(u, v\) to a chart by choosing \(n - 2\) coordinates \(x^i\) independent of \(u\) in \(\Omega\).

- **Expression of** \(g\) **in a Brinkmann chart**:

\[
g = -2du \left( dv + H(u, x^k)du + W_i(u, x^k)dx^i \right) + g_{ij}(u, x^k)dx^i dx^j
\]

(natural sum in repeated indexes, \(K \equiv -\partial_v\))
Construction of a Brinkmann chart

- **Brinkmann chart** \( \{u, v, x^i\} \): complete \( u, v \) to a chart by choosing \( n - 2 \) coordinates \( x^i \) independent of \( u \) in \( \Omega \).

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(natural sum in repeated indexes, \( K \equiv -\partial_v \))

**Remark**

Being more careful, one could get \( H = 0 \) and \( W_i = 0 \)!

But it is preferred as above, as we wish to remove the \( u \)-dependence of \( g_{ij}(u, x^i) \).
In general:

*Study of degenerate hypersurfaces*

$\leadsto$ *Transverse vector field* $\xi$

Non-unique $\xi$: wise choice when possible.
Geometric developments

- In general:

  *Study of degenerate hypersurfaces*
  
  \( \rightsquigarrow \) *Transverse vector field \( \xi \)*

  Non-unique \( \xi \): wise choice when possible.

- This happens in Brinkmann spaces too:

  *degenerate hypersurfaces \( \Sigma_u \) with transverse \( \partial_u \)*
  
  *(non-univocally determined)*
Geometric developments

- In general:

  *Study of degenerate hypersurfaces*
  
  ~~~~~
  Transverse vector field $\xi$
  ~~~~~

  Non-unique $\xi$: wise choice when possible.

- This happens in Brinkmann spaces too:

  *degenerate hypersurfaces $\Sigma_u$ with transverse $\partial_u$*
  
  *(non-univocally determined)*

- Issues on Brinkmann spaces:
  
  - Relations between different choices of $\partial_u$ (and $\Omega$)
  - To introduce associated geometric objects with nice properties
  - Study potentially extensible to other degenerate cases
Geometric developments

- **Foliations**
  1. Spacelike \((n-2)\)-foliation \(\mathcal{M}: \{u = u_0, \nu = \nu_0\}\)
  2. Timelike 2 foliation: \(\mathcal{U}: \{x^i = x_0^i\}\)
Geometric developments

- **Foliations**
  1. Spacelike \((n-2)\)-foliation \(\mathcal{M} : \{u = u_0, v = v_0\}\)
  2. Timelike 2 foliation: \(\mathcal{U} : \{x^i = x^i_0\}\)

- **Tangent bundle decompositions**:
  1. Non-orthogonal: \(TM = T\mathcal{M} \oplus TU\)
  2. Orthogonal: \(TM = TU \oplus (TU)\perp\)

- **Natural bases**:
  1. \(TU = \text{span}\{E_0 := \partial_u - H\partial_v, E_1 := \partial_v\}\)
  2. \((TU)\perp = \text{span}\{E_i := \partial_i - W_i\partial_v\}\)
  3. \(T\mathcal{M} = \text{span}\{\partial_i\}\)
The spacelike foliation $\mathcal{M}$

Foliation $\mathcal{M}$: \{\(u = u_0, \nu = \nu_0\}\}

Metric induced bundle by the foliation:

\[
\bar{g} = g_{ij} \bar{dx}^i \bar{dx}^j
\]

(Notation: if \(dx^i, \alpha\) on \(M\), then \(\bar{dx}^i, \bar{\alpha}\) on the foliation)
For any 1-form $\alpha$ on $M$:

$$\overline{d} \overline{\alpha} = \overline{d\alpha}.$$  

Satisfies the properties of a derivation for $\omega, \tau \in \Lambda^q M$:

1. Linearity plus $\overline{d}(\omega \wedge \tau) = \overline{d}\omega \wedge \tau + (-1)^s \omega \wedge \overline{d}\tau$.
2. $\overline{d}(\overline{d}\omega) = 0$.
3. If $\omega = \frac{1}{s!} \omega_{i_1...i_s} \overline{d}x^{i_1} \wedge \ldots \overline{d}x^{i_s}$, then
   $$\overline{d}\omega = \frac{1}{s!} \partial_k (\omega_{i_1...i_s}) \overline{d}x^k \wedge \overline{d}x^{i_1} \wedge \ldots \overline{d}x^{i_s}$$
4. Poincaré Lemma: $\overline{d}$-closed implies $\overline{d}$-exact.
Covariant derivative $\bar{\nabla}$ for $\mathcal{M}$

- Vector fields on $\mathcal{M}$ are naturally on $\mathcal{M}$
- $\mathcal{M}$ is endowed with a Riemannian metric and then a natural $\bar{\nabla}$

\[ \bar{\nabla}_X Y \in \mathfrak{X}(\mathcal{M}) \quad \forall X, Y \in \mathfrak{X}(\mathcal{M}) \]

Extended to tensor fields on $\mathcal{M}$ satisfies

\[ \bar{\nabla}g = 0 \]
Covariant derivative $\overline{\nabla}$ for $\mathcal{M}$

- Vector fields on $\mathcal{M}$ are naturally on $\mathcal{M}$
- $\mathcal{M}$ is endowed with a Riemannian metric and then a natural $\overline{\nabla}$

$$\overline{\nabla}_X Y (\in \mathfrak{X}(\mathcal{M})) \quad \forall X, Y \in \mathfrak{X}(\mathcal{M})$$

Extended to tensor fields on $\mathcal{M}$ satisfies

$$\overline{\nabla} g = 0$$

Defines a foliation curvature $\overline{\mathcal{R}}$:

$$\overline{\mathcal{R}}(X, Y) Z = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]} )Z \in \mathfrak{X}(\mathcal{M}), \ \forall X, Y, Z \in \mathfrak{X}(\mathcal{M})$$

plus Ricci tensor $\overline{\mathcal{Ric}}$ and scalar curvature $\overline{S}$. 

M. Sánchez
Lorentzian $r$-th symmetric spaces
Covariant derivative $\nabla$ for $\mathcal{M}$

**Definition**

- $\mathcal{M}$ is **flat** (resp. **locally symmetric**) if $\overline{\mathcal{R}} = 0$ (resp. $\overline{\nabla} \overline{\mathcal{R}} = 0$)
- $u$-**Einstein** if $\overline{\mathcal{R}}ic = \mu \overline{g}$ for some $\mu$ s.t. $d\mu \wedge du = 0$ (Schur lemma $\mathcal{R}ic = fg \Rightarrow f \equiv c$ does not apply to foliations) and:
  1. $\mathcal{M}$ is **Einstein** if $\mu = \text{const.}$,
  2. $\mathcal{M}$ is **Ricci-flat** if $\mu \equiv 0$. 

**Understanding Brinkmann**

Adapted geometric elements

Reducibility and Eisenhart thm
Covariant derivative $\bar{\nabla}$ for $\mathcal{M}$

From Riemannian results:

**Proposition**

Let $(\mathcal{M}, g)$ be a Brinkmann space:

1. $\bar{\nabla}^r \mathcal{R} = 0$ (rth-symmetric) $\implies$ $\bar{\nabla} \mathcal{R} = 0$ (locally symmetric).
2. $\bar{\nabla} \mathcal{R} = 0$ (locally symmetric) and $\bar{\mathcal{Ric}} = 0$ (Ricci-flat) $\implies$ $\mathcal{R} = 0$ (flat).
3. If $\mathcal{M}$ is flat, there exists a chart \{u, v, y^i\} s.t.:
   
   \[ g = -2du(dv + Hdu + W_i dy^i) + \delta_{ij} dy^i dy^j. \]
   
   ($g_{ij} = \delta_{ij}$ independent of u)
Transverse operators for $\mathcal{M}$: dot derivative

For $T \in \Gamma(\mathcal{T}_s \mathcal{M})$:

$$\dot{T} = \mathcal{L}_{\partial_u} T \in \Gamma(\mathcal{T}_s \mathcal{M})$$

That is, in the base $\{\partial_i\}$:

$$\dot{T}^{i_1\ldots i_r}_{j_1\ldots j_s} = \partial_u (T^{i_1\ldots i_r}_{j_1\ldots j_s})$$
Transverse operators for $\mathcal{M}$: $D_0$ derivative

Recall $E_0 = \partial_u - H \partial_v$

\[ D_0 : \Gamma(T^r_s \mathcal{M}) \rightarrow \Gamma(T^r_s \mathcal{M}) \]

\[ T \rightarrow D_0 T = (\nabla_{E_0} \bar{T}) \]
Transverse operators for $\mathcal{M}$: $D_0$ derivative

Recall $E_0 = \partial_u - H \partial_v$

$$D_0 : \Gamma(T^r_s \mathcal{M}) \rightarrow \Gamma(T^r_s \mathcal{M})$$

$$T \rightarrow D_0 T = (\nabla_{E_0} \tilde{T})$$

Properties:

1. Algebraic properties of a tensor derivation
2. $D_0 \bar{g} = 0$

Lemma

*Each vector field on a leaf of $\mathcal{M}$ can be extended to a unique $K(= -\partial_v)$-invariant $D_0$-parallel vector field in $\mathfrak{X}(\mathcal{M})$.***
Reducibility in $\mathcal{M}$

$T \in \Gamma(T^k_s\mathcal{M})$ is reducible if, there are foliations $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ s.t., in a natural sense:

$$TM = TM^{(1)} \oplus TM^{(2)} \quad T = T^{(1)} \oplus T^{(2)}$$

i.e. there exists a Brinkmann chart $\{u, v, x^i\}$ and a partition of the indexes $I_1 = \{2, \ldots, d + 1\}$, $I_2 = \{d + 2, \ldots, n - 1\}$ s.t.

$$T_{aa'} = 0 \quad \partial_{a'} T_{ab} = 0,$$

where $a, b$ belong to some $I_m$ and $a', b'$ to the other one.
Reducibility in $\mathcal{M}$

$T \in \Gamma(T^k_s \mathcal{M})$ is reducible if, there are foliations $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ s.t., in a natural sense:

$$T\mathcal{M} = T\mathcal{M}^{(1)} \oplus T\mathcal{M}^{(2)} \quad T = T^{(1)} \oplus T^{(2)}$$

i.e. there exists a Brinkmann chart $\{u, v, x^i\}$ and a partition of the indexes $I_1 = \{2, \ldots, d + 1\}, I_2 = \{d + 2, \ldots, n - 1\}$ s.t.

$$T_{aa'} = 0 \quad \partial_{a'} T_{ab} = 0,$$

where $a, b$ belong to some $I_m$ and $a', b'$ to the other one.

In particular, when $\bar{g} \in \Gamma(T_2 \mathcal{M})$ is reducible the sum is orthogonal and we write $\mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$, 

$$g = -2 du (dv + H du + \hat{W}) + \dot{g}^{(1)} \oplus \dot{g}^{(2)}$$
Extended Eisenhart theorem

**Theorem**

Let $(M, g)$ be a Brinkmann space and $\{u, v, x^i\}$ a Brinkmann chart. If there exist a symmetric $\overline{L} \in \Gamma(T^0_2 M)$, $\overline{L} \neq c\bar{g}$, which is $v$-invariant, $\overline{\nabla}$-parallel and $D_0$-parallel.
Theorem

Let \((M, g)\) be a Brinkmann space and \(\{u, v, x^i\}\) a Brinkmann chart. If there exist a symmetric \(\bar{L} \in \Gamma(T_2^0 M)\), \(\bar{L} \neq c\bar{g}\), which is \(v\)-invariant, \(\bar{\nabla}\)-parallel and \(D_0\)-parallel.

Then there exists a Brinkmann chart \(\{u, v, y^i\}\) in the Brinkmann decomposition \(\{u, v\}\) such that:

1. \(\bar{g}\) is reducible: 
   \[
   \bar{g} = \bar{g}^{(1)} \oplus \ldots \oplus \bar{g}^{(s)}, \quad s \geq 2 \quad (u\text{-dependent})
   \]

2. \(\bar{L} = \sum_{m=1}^{s} \lambda_m \bar{g}^{(m)}\) for some \(\lambda_m \in \mathbb{R}\) \((u\text{-independent}, \lambda_m = 0)\).
Local version of the theorem

Aim:

Theorem

A properly 2nd-symmetric Brinkmann space is locally isometric to a product of:

- a proper 2nd-order Cahen-Wallach space \((\mathbb{R}^{d+2}, g_A)\),
  \[
g_A = -2 du \left( dv + (a_{ij} u + b_{ij}) x^i x^j du \right) + \delta_{ij} dx^i dx^j
  \]
  with some \(a_{ij} \neq 0\), and

- symmetric Riemannian space \((N, g_N)\).
Step 1: define appropriate elements on $\mathcal{M}$

Express the non-trivial parts of $R, \nabla R$ in terms of tensors on $\mathcal{M}$

- Tensors for $R$: $A \in T_2 \mathcal{M}, B \in T_3 \mathcal{M}, \bar{R} \in T_3^1 \mathcal{M}$
  - $A(X, Y) = \theta^1(R(E_0, \dot{Y})\dot{X}), \text{ i.e. } A_{ij} = R^1_{\ i0j}$
  - $B(X, Y, Z) = \theta^1(R(\dot{Y}, \dot{Z})\dot{X}), \text{ i.e. } B_{ijk} = R^1_{\ ijk}$
  - $\bar{R}(X, Y)Z = \bar{R}(\dot{X}, \dot{Y})\dot{Z}, \text{ i.e. } \bar{R}^i_{\ jkl} = R^i_{\ jkl}$

- Tensors for $\nabla R$: $\tilde{A} \in T_2 \mathcal{M}, \hat{A}, \tilde{B} \in T_3 \mathcal{M}, \hat{B}, \tilde{R} \in T_3^1 \mathcal{M}$
  - $\tilde{A}(X, Y) = \theta^1\left((\nabla_{E_0}R)(E_0, \dot{Y})\dot{X}\right), \hat{A}(X, Y, Z) = \theta^1\left((\nabla_{\dot{X}}R)(E_0, \dot{Z})\dot{Y}\right)$
  - $\tilde{B}(X, Y, Z) = \theta^1\left((\nabla_{E_0}R)(\dot{Y}, \dot{Z})\dot{X}\right), \hat{B}(X, Y, Z, V) = \theta^1\left((\nabla_{\dot{X}}R)(\dot{Z}, \dot{V})\dot{Y}\right)$
  - $\tilde{R}(X, Y)Z = \nabla_{E_0}R(\dot{X}, \dot{Y})\dot{Z}$.

\[
\tilde{A}_{ij} = \nabla_{0} R^1_{\ i0j}; \quad \hat{A}_{sij} = \nabla_{s} R^1_{\ i0j} \\
\tilde{B}_{ijk} = \nabla_{0} R^1_{\ ijk}; \quad \hat{B}_{sijk} = \nabla_{s} R^1_{\ ijk}; \quad \tilde{R}^{i}_{\ jkl} = \nabla_{0} R^{i}_{\ jkl}
\]
Step 2: simplification of chart-dependent elements

Proposition

For any 2nd-symmetric Brinkmann decomposition \( \{u, v\} \):

(a) All the (chart-dependent) elements for \( \nabla R \) vanish but \( \tilde{A} \), i.e.
\[
\tilde{B} = \tilde{R} = \tilde{A} = \tilde{B} = 0.
\]

(b) \( \tilde{A} \) is independent of the chosen chart

(c) The equations of 2nd symmetry reduce to:
\[
\begin{align*}
\nabla \tilde{A} &= 0, & D_0 \tilde{A} &= 0 \\
\nabla \tilde{R} &= 0, & D_0 \tilde{R} &= 0
\end{align*}
\]

with \( \tilde{B} = 0, \tilde{\dot{B}} = 0, \tilde{\dot{A}} = 0 \).
Step 2: simplification of chart-dependent elements

*Ingredients of the proof.* A first simplification comes from
\[ \nabla^2 R = 0 \Rightarrow \nabla R = 0. \]
Step 2: simplification of chart-dependent elements

**Ingredients of the proof.** A first simplification comes from $\nabla^2 R = 0 \Rightarrow \nabla R = 0$. Then:

- Use the conditions of integrability of 2nd symmetry equations

\[
(\nabla_k D_0 - D_0 \nabla_k) F^i{}_j = (H, k)(\partial_v F^i{}_j) + F^i{}_m B_{kj}^m - F^m{}_j B_{km}^i - t^m{}_k \nabla_m F^i{}_j
\]

\[
(\nabla_n \nabla_m - \nabla_m \nabla_n) T^{i_1 \ldots i_k}_{j_1 \ldots j_s} = \sum_{b=1}^s \tilde{R}^l{}_{j_b n m} T^{i_1 \ldots i_k}_{j_1 \ldots j_{b-1} j_{b+1} \ldots j_s} - \sum_{a=1}^k \tilde{R}^i{}_{l n m} T^{i_1 \ldots i_{a-1} i_{a+1} \ldots i_k}_{j_1 \ldots j_s}
\]
Step 2: simplification of chart-dependent elements

*Ingredients of the proof.* A first simplification comes from \( \overline{\nabla}^2 R = 0 \Rightarrow \overline{\nabla} R = 0. \) Then:

- Use the conditions of integrability of 2nd symmetry equations

\[
(\overline{\nabla}_k D_0 - D_0 \overline{\nabla}_k) F^i \ j = (H_k) (\partial_v F^i \ j) + F^i \ m B_{kj}^m - F^m \ j B_{km}^i - t^m \ k \overline{\nabla}_m F^i \ j
\]

\[
(\overline{\nabla}_n \overline{\nabla}_m - \overline{\nabla}_m \overline{\nabla}_n) T_{j_1 \ldots j_s}^{i_1 \ldots i_k} = \sum_{b=1}^{s} \overline{R}'_{jb nm} T_{j_1 \ldots j_{b-1} j_{b+1} \ldots j_s}^{i_1 \ldots i_k} - \sum_{a=1}^{k} \overline{R}'_{ia} l m T_{j_1 \ldots j_s}^{i_1 \ldots i_{a-1} i_{a+1} \ldots i_k}
\]

- Use the equations derived from 2nd Bianchi identity

\[
\nabla [\alpha R_{\beta \lambda \nu \mu}] = 0 \Rightarrow \tilde{R}_{ijkl} = -2 \tilde{B}_{ijkl}, \quad \tilde{B}_{kij} = 2 \tilde{A}_{[ij]k}.
\]

Technical point: algebraic criteria for the vanishing of tensor fields are also introduced, as:

In an Euclidean vector space, \( T_{ijk} \) vanishes if

\[
T_{i[jk]} = T_{ijk}, \quad T_{ijk} + T_{jki} + T_{kij} = 0 \quad \text{and} \quad T_{(ij)} \ T_{rnm} = 0
\]
Step 2: simplification of chart-dependent elements

Remark

- $\nabla R \neq 0$ iff $\tilde{A} \neq 0$.
- The scalar curvature $S$ (not only of $\mathcal{M}$ but also $\tilde{M}$) of $M$ is constant.
Step 3: Reducibility of $\tilde{A}$ and $\tilde{\text{Ric}}$

From the equations of 2nd-symmetry:

\[
\nabla \tilde{A} = 0, \quad D_0 \tilde{A} = 0
\]
\[
\nabla \tilde{R} = 0, \quad D_0 \tilde{R} = 0
\]

$\tilde{A}$ and $\tilde{\text{Ric}}$ (and also $g$) are $D_0$-$\nabla$-invariant so that Extended Eisenhart theorem applies and:
Step 3: Reducibility of \( \tilde{A} \) and \( \bar{\text{Ric}} \)

- \( \mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)} \) with \( \mathcal{M}^{(1)} \) flat and \( \mathcal{M}^{(2)} \) locally symmetric non Ricci-flat.
Step 3: Reducibility of $\tilde{\mathcal{A}}$ and $\tilde{\text{Ric}}$

- $\mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$ with $\mathcal{M}^{(1)}$ flat and $\mathcal{M}^{(2)}$ locally symmetric non Ricci-flat.
- $\tilde{g} = \tilde{g}^{(1)} \oplus \tilde{g}^{(2)}$ with $\tilde{g}^{(1)} = \delta_{ab}dx^a dx^b$ ($\dot{g}^{(1)} = 0$, i.e., $u$-independent)
Step 3: Reducibility of $\tilde{A}$ and $\overline{\text{Ric}}$

- $\mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$ with $\mathcal{M}^{(1)}$ flat and $\mathcal{M}^{(2)}$ locally symmetric non Ricci-flat.
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- $\overline{R} = \overline{R}^{(1)} \oplus \overline{R}^{(2)}$ with $\overline{R}^{(1)} = 0$ and $\overline{R}^{(2)} \neq 0$ with $\overline{\nabla} \overline{R}^{(2)} = 0$. 

Remark
For any Brinkmann decomposition $\{u, v\}$: $\tilde{A}$, $\text{Ric}$, and $g$ are simultaneously reducible.
The non-trivial part of $\tilde{A}$ lies in $\mathcal{M}^{(1)}$ and the non-trivial one of Ricci on $\mathcal{M}^{(2)}$. 

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Lorentzian r-th symmetric spaces
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- $\bar{R} = \bar{R}^{(1)} \oplus \bar{R}^{(2)}$ with $\bar{R}^{(1)} = 0$ and $\bar{R}^{(2)} \neq 0$ with $\nabla \bar{R}^{(2)} = 0$.
- $\tilde{A} = \tilde{A}^{(1)} \oplus \tilde{A}^{(2)}$ with $\tilde{A}^{(2)} = 0$. 

Remark: For any Brinkmann decomposition $\{u, v\}$: $\tilde{A}$, $\text{Ric}$ and $g$ are simultaneously reducible. The non-trivial part of $\tilde{A}$ lies in $\mathcal{M}^{(1)}$ and the non-trivial one of Ricci on $\mathcal{M}^{(2)}$. 

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Remark

For any Brinkmann decomposition $\{u, v\}$:
- $\tilde{A}$, $\overline{\text{Ric}}$ and $\overline{g}$ are simultaneously reducible
- The non-trivial part of $\tilde{A}$ lies in $\mathcal{M}^{(1)}$ and the non-trivial one of Ricci on $\mathcal{M}^{(2)}$
Step 4: reduction to two independent Lorentzian problems

From previous result in a Brinkmann chart:

\[ g = -2du(dv + H du + \hat{W}) + \hat{g}^{(1)} \oplus \hat{g}^{(2)} \]

and one can check that \( H, \ W \) are also simultaneously reducible, so that in some new chart:

\[ g = -2du(dv + (H^{(1)} + H^{(2)}) du + \hat{W}^{(1)} + \hat{W}^{(2)}) + \hat{g}^{(1)} \oplus \hat{g}^{(2)} \]
Step 4: reduction to two independent Lorentzian problems

Now, define two lower dimensional Lorentzian spaces $M^m = \mathbb{R}^2 \times \overline{M}^m$, $m = 1, 2$:

$$g^m = -2du(dv + H^m du + W^m) + \bar{g}^m.$$ 

Remark

- These two Lorentzian spaces are 2nd symmetric as so was the original one.
- So, the problem is reduced to the 2nd symmetry of two simple spaces.
Step 4: reduction to two independent Lorentzian problems

\((M^2, g^2)\) 2nd symmetric with \(\tilde{A}^2 = 0:\)

- Locally symmetric
- Cahen-Wallach space (order 1) compatible with parallel \(K = -\partial_v\) (and \(A^{[2]} = 0\))
Step 4: reduction to two independent Lorentzian problems

- \((M^{[2]}, g^{[2]})\) 2nd symmetric with \(\tilde{A}^{[2]} = 0\):
  - Locally symmetric
  - Cahen-Wallach space (order 1) compatible with parallel \(K = -\partial_v\) (and \(A^{[2]} = 0\))

\[\rightsquigarrow\] Locally symmetric Riemannian part in Thm
Step 4: reduction to two independent Lorentzian problems

- $(M^{[2]}, g^{[2]})$ 2nd symmetric with $\tilde{A}^{[2]} = 0$:
  - Locally symmetric
  - Cahen-Wallach space (order 1) compatible with parallel $K = -\partial_v$ (and $A^{[2]} = 0$)

  $\leadsto$ Locally symmetric Riemannian part in Thm

- $(M^{[1]}, g^{[1]})$ 2nd-symmetric with flat $M^{[1]}$ ($\tilde{A}^{[1]} \neq 0$):
  2nd-symmetric plane wave:
Step 4: reduction to two independent Lorentzian problems

- \((M[2], g[2])\) 2nd symmetric with \(\tilde{A}[2] = 0\):
  - Locally symmetric
  - Cahen-Wallach space (order 1) compatible with parallel
    \(K = -\partial_v\) (and \(A[2] = 0\))

  \[\sim\] Locally symmetric Riemannian part in Thm

- \((M[1], g[1])\) 2nd-symmetric with flat \(M[1]\) \((\tilde{A}[1] \neq 0)\):
  2nd-symmetric plane wave: directly computable obtaining a
generalized Cahen-Wallach of order 2:

\[
g_A = -2du \left( dv + (a_{ij}u + b_{ij})x^i x^j du \right) + \delta_{ij} dx^i dx^j
\]
Further open questions

Modest:

1. Characterize accurately when $\nabla^2 T = 0 \nRightarrow \nabla T = 0$ in the Lorentzian case.

2. Classify 3rd symmetric Lorentzian spaces.
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Ambitious:

1. Generalize to Lorentzian $r$th-symmetric spaces
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2. Idem to higher signatures.

Senovilla’s:

1. Solve all the linear conditions for curvature:
$$\nabla^r R + t_1 \otimes \nabla^{r-1} R + t_2 \otimes \nabla^{r-2} R + \cdots + t_{r-1} \otimes \nabla R + t_r \otimes R = 0$$
for some $m$-covariant tensors $t_m$. 