From constant mean curvature surfaces to overdetermined elliptic problems

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The problem:

To classify domains $\Omega \in \mathbb{R}^n$ that support a positive solution of the over-determined elliptic system

$$\begin{align*}
\Delta u + f(u) &= 0 \quad \text{in} \quad \Omega \\
 u &= 0 \quad \text{on} \quad \partial \Omega \\
\frac{\partial u}{\partial \nu} &= \text{constant} \quad \text{on} \quad \partial \Omega
\end{align*}$$
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To classify domains $\Omega \in \mathbb{R}^n$ that support a positive solution of the over-determined elliptic system

$$\begin{cases}
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\end{cases}$$

Theorem (Serrin, 1971). If $\Omega$ is bounded it is a ball
The non compact case
The non compact case

The problem becomes: to classify unbounded domains $\Omega \in \mathbb{R}^n$ that support a positive solution of the over-determined elliptic system

$$
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\end{cases}
\]

**Definition:** If such problem is solvable, \( \Omega \) is a \( f \)-extremal domain
Constant mean curvature surfaces

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*Recall: the mean curvature $H(p)$ in a point $p$ of a given hypersurface is the sum (or the mean) of the principal curvatures at $p$.***
Conjecture of Berestycki, Caffarelli and Nirenberg


\[
\begin{aligned}
\Delta u + f(u) &= 0 \quad \text{in} \quad \Omega \\
u &> 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega \\
\frac{\partial u}{\partial \nu} &= \text{constant} \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

EXTRA HYPOTHESIS

\[\mathbb{R}^n \setminus \overline{\Omega} \text{ connected} \]

\[u \text{ bounded}\]

\[\downarrow\]

\[\Omega \text{ is a half space, or a ball, or a cylinder } \mathbb{R}^j \times B \text{ (where } B \text{ is a ball) or the complement of one of these three exemples}.\]
Many results from EDP’s community
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Reichel (Arch. Rat. Mech. & An.)

Some rigidity results for exterior domains, for some very special kind of functions $f$, with some special behaviours of the solution $u$ at infinity
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Some rigidity results for epigraphs, for some very special kind of functions $f$, with some special assumptions of asymptotical flatness for the boundary of the domain.
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**Farina-Valdinoci** (*Arch. Rat. Mech. & An.*)

Some rigidity results for epigraphs in \( \mathbb{R}^2 \) for all functions \( f \), and in \( \mathbb{R}^3 \) for some classes of functions \( f \).
Coming back to constant mean curvature surfaces

In $\mathbb{R}^n$ there are non compact surfaces with constant mean curvature! For exemple the Delaunay surfaces...
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It is a one parameter smooth family of constant mean curvature surfaces in $\mathbb{R}^3$. 
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It is a one parameter smooth family of constant mean curvature surfaces in $\mathbb{R}^3$.

They are periodic perturbations of a cylinder and are surfaces of revolution.
A parallel result on overdetermined elliptic problems

**Theorem (S. 2010 & Schlenk-S. 2011):** For $n \geq 2$ there exists a smooth family of periodic perturbations $\Omega$ of the cylinder $B^{n-1} \times \mathbb{R}$ ($B^{n-1}$ = unit ball), with boundary of revolution, where there exists a periodic and positive solution to

$$\begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial\Omega \\
\frac{\partial u}{\partial \nu} &= \text{constant} \quad \text{on} \quad \partial\Omega
\end{align*}$$
About the conjecture
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The result is true also in dimension 2. But this is not a counterexample to the conjecture.

The complement of such domain is not connected.
The conjecture in dimension 2
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**Theorem (Ros-S. 2013)**

The conjecture of Berestycki-Caffarelli-Nirenberg in dimension 2 is true for all function $f$ such that $f(t) \geq \lambda t$ for a $\lambda > 0$. 

The proof of the theorem (1)

**Step 1.** If $\mathbb{R}^2 \setminus \overline{\Omega}$ is connected then there are only three possibilities for $\Omega$:

1. $\Omega$ is bounded (Done by Serrin !)
2. $\Omega$ is an exterior domain (Easy case, EDP techniques)
3. $\partial \Omega$ is an open curve that separated $\mathbb{R}^2$ in two connected components, and $\Omega$ is one of such components (Hard case)

Definition. A surface has finite topology if

1. it is a compact surface
2. outside of a big ball, the surface is done of a finite number of noncompact components diffeomorphic to $S^{n-1} \times \mathbb{R}_+$, called ends.

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Theorem. Let $S$ be a properly embedded finite topology nonzero CMC surface in $\mathbb{R}^3$. Then $S$ cannot have only one end.
The proof of the theorem

**Definition.** We say that a domain has finite topology if

1. it is bounded domain, or
2. it is the complement of a compact domain, or
3. outside of a big ball, the domain is done of a finite number of noncompact components diffeomorphic to $B^{n-1} \times \mathbb{R}_+$, called ends.
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**Proposition (Ros-S.).** If $f(t) \geq \lambda t$ for some $\lambda > 0$ and $\Omega$ is an $f$-extremal domain, then $\Omega$ cannot have only one end.
Theorem: Korevaar, Kusner & Solomon, 1989

Let $S$ be a properly embedded finite topology nonzero CMC surface in $\mathbb{R}^3$ contained in a cylinder. Then $S$ is surface of revolution.
An other result

**Theorem (Ros-S. 2013).** Let $\Omega$ be an $f$-extremal domain of $\mathbb{R}^2$ with bounded curvature. If $\Omega$ is contained in a half-plane, then $\Omega$ is either a ball or a half-plane or there exists a positive function $\varphi : \mathbb{R} \rightarrow ]0, \infty[$ such that $\Omega$ is $\{|y| < \varphi(x)\}$.

**Corollary.** Proof of the conjecture of Berestycki-Caffarelli-Nirenberg in the half-plane under the assumption of bounded curvature.
Generalization to $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ (Morabito-S.)

Existence of Delaunay type domains with a positive solution of

\[
\begin{aligned}
\Delta u + \lambda u &= 0 & \text{in} & \quad \Omega \\
u &= 0 & \text{on} & \quad \partial\Omega \\
\frac{\partial u}{\partial \nu} &= \text{constant} & \text{on} & \quad \partial\Omega
\end{aligned}
\]
Open problems and conjectures (Ros-S.)
Basic conjecture:

A class of CMC hypersurfaces in $\mathbb{R}^{n+1}$ (which?)<---domains in $\mathbb{R}^n$ that support a positive solution to the problem

$$
\begin{aligned}
\Delta u + f(u) &= 0 \quad \text{in} \quad \Omega \\
    u &= 0 \quad \text{on} \quad \partial\Omega \\
    \frac{\partial u}{\partial \nu} &= \text{const} \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

for $f(t) = \lambda t$ (and maybe others?)
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1. Existence of "THE" family of Delaunay type domains for overdetermined problems, from the cylinder to the sphere (as limit).
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2. Rigidity of the ends of overdetermined domains: all the ends have the asymptotic of such Delaunay domains.
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3. Gluing method: existence of highly nontrivial overdetermined domains with such kind of ends.
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2. Rigidity of the ends of overdetermined domains: all the ends have the asymptotic of such Delaunay domains.

3. Gluing method: existence of highly nontrivial overdetermined domains with such kind of ends.

4. Correspondence between some kind of minimal surfaces and harmonic overdetermined problems.
Classification of harmonic overdetermined solutions

**Theorem (Traizet 2013).**

Domains in $\mathbb{R}^2$ that support a positive solution to the problem

$$
\begin{align*}
\Delta u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega \\
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\end{align*}
$$

with the hypothesis that $\partial \Omega$ has a finite number of components, at least in the quotient if $\Omega$ is periodic.
The Scherk simply periodic minimal surface
The Scherk type domain
The Scherk type domain
The Scherk type domain

Such domain was found by the physicians Baker, Saffman and Sheffield in 1979 as a solution to an equilibrium problem in hydrodynamics of vortices!
We give the idea of the proof of:

**Theorem (Ros-S. 2013).** Let $\Omega$ be an $f$-extremal domain of $\mathbb{R}^2$ with bounded curvature. If $\Omega$ is contained in a half-plane, then $\Omega$ is either a ball or a half-plane or there exists a positive function $\varphi : \mathbb{R} \rightarrow ]0, \infty[$ such that $\Omega$ is the symmetric domain $\{|y| < \varphi(x)\}$.
Step 1: The moving plane argument
Step 2: The tilted moving plane argument

\[ \Omega_1 \supset \Omega_2 \]

\[ \Sigma(T) \supset \Sigma'(T) \]

\[ p \]

\[ p' \]

\[ y = -\epsilon x + a \]
Step 3: The implication of the moving plane argument
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Step 4: From classic PDE’s theory
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Bounded curvature of $\partial \Omega$ implies that $\nabla u$ is bounded.
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