Capacity, number of ends and asymptotic planes in minimal submanifolds

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Outline

1 Cheeger isoperimetric constant and fundamental tone
   - Cheeger constant
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   - extrinsic distance and extrinsic balls
   - finite volume growth
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   - Number of ends

2 Capacity
Cheeger isoperimetric constant I

Given $M^n$ a complete and non compact Riemannian manifold of dimension greater than 1 ($n \geq 2$), the Cheeger isoperimetric constant is defined by this quotient

$$I_\infty(M) := \inf_{\Omega \subset M} \frac{\text{Vol}_{n-1}(\partial \Omega)}{\text{Vol}_n(\Omega)}.$$  \hspace{1cm} (1)

where $\Omega$ ranges over compact open subsets $\Omega \subset M$ with smooth boundaries $\partial \Omega$.

Cheeger constant examples

- $I_\infty(\mathbb{R}^n) = 0$.
- $I_\infty(\mathbb{H}^n(b)) = (n - 1) \sqrt{-b}$. 
The fundamental tone $\lambda^*(M)$ of a smooth Riemannian manifold $M$ is defined by the infimum of the quotient between the squared norm of the gradient and the squared norm of functions

$$\lambda^*(M) = \inf_{f \in L^2_{1,0}(M) \setminus \{0\}} \left\{ \frac{\int_M |\nabla f|^2 d\mu}{\int_M f^2 d\mu} \right\}$$

(2)

where the functions ranges in $L^2_{1,0}(M)$, the completion of smooth functions with compact support $C^\infty_0(M)$ with respect to this norm

$$\|\phi\|^2 = \int_M \phi^2 d\mu + \int_M |\nabla \phi|^2 d\mu$$
**Theorem (Cheeger)**

Let $M$ be a complete non compact manifold, then the Cheeger isoperimetric constant is a bound for the fundamental tone

$$\lambda^*(M) \geq \frac{I_\infty(M)^2}{4}$$ (3)

And for minimal submanifolds of the Hyperbolic space

**Corollary (S-T Yau, McKean, Chavel)**

Let $M^n \hookrightarrow \mathbb{H}^m(b)$ be a complete, minimally immersed submanifold of $\mathbb{H}^m(b)$, then the Cheeger constant (and so the fundamental tone) are bounded from below by the following expressions

$$I_\infty(M) \geq (n - 1)\sqrt{-b},$$

$$\lambda^*(M) \geq \frac{-(n - 1)^2b}{4}.$$ (4)
Corollary

Let $M^n \hookrightarrow N$ be a complete, minimally immersed submanifold of a Cartan-Hadamard manifold $N$ (simply connected with sectional curvatures $K_N$ bounded above by $K_N \leq b \leq 0$), then the Cheeger constant (and so the fundamental tone) are bounded from below by the following expressions

$$I_\infty(M) \geq (n-1)\sqrt{-b},$$

$$\lambda^*(M) \geq \frac{-(n-1)^2b}{4}.$$  \hspace{1cm} (5)
proof 1

By the expression of the Hessian for submanifolds and the Hessian comparisons given by Greene-Wu for the extrinsic distance function

\[ \Delta^M r \geq (n - 1) \cot_b(r), \]

being

\[ \cot_b(r) = \begin{cases} \frac{1}{r} & \text{if } b = 0, \\ \sqrt{-b} \coth(\sqrt{-br}) & \text{if } b < 0 \end{cases} \]

Therefore,

\[ \Delta^M r \geq (n - 1) \sqrt{-b}, \]

Integrating on \( \Omega \subset M \)

\[ \int_{\Omega} \Delta^M r dV \geq (n - 1) \sqrt{-b} \Vol_n(\Omega), \]
By the divergence theorem

\[ \int_{\partial \Omega} \langle \nabla r, \nu \rangle dA \geq (m - 1) \sqrt{-b} \text{Vol}_n(\Omega), \]

Hence,

\[ \text{Vol}_{n-1}(\partial \Omega) \geq (n - 1) \sqrt{-b} \text{Vol}_n(\Omega), \]
¿what was known? I

Theorem A. Candel, Transactions AMS, 2007

Let $M$ be a complete simply connected stable minimal surface in the hyperbolic space $\mathbb{H}^3(-1)$, then

$$\frac{1}{4} \leq \lambda^*(M) \leq \frac{3}{4}.$$ 

Theorem A. Candel, Transactions AMS, 2007

The fundamental tone of the minimal catenoids (given in Do Carmo - Dajczer, Rotation hypersurfaces in spaces of constant curvature. Trans. Amer. Math. Soc. ,1983) in the hyperbolic space $\mathbb{H}^3(-1)$ is

$$\lambda^*(M) = \frac{1}{4}.$$
¿what was known? II

The minimal catenoids satisfy

$$\int_M |A|^2 d\mu < \infty .$$

(6)
Let $M^n$ be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}(-1)$ with $\int_M |A|^2 d\mu < \infty$. Then we have

$$\frac{(n-1)^2}{4} \leq \lambda^*(M) \leq n^2.$$  

(7)
Corollary V. Gimeno (REAG-ICMAT 2012)

Given a complete submanifold $M^n \hookrightarrow N$ properly and minimaly immersed in a Cartan Hadamard $N$ ambient manifold with sectional curvatures $K_N$ bounded above $K_N \leq b \leq 0$, suppose moreover that the immersion has finite volume growth. Then, we obtain the following upper bound for the fundamental tone of the submanifold

$$\lambda^*(M) \leq 4I_\infty^2(M) = -4(n - 1)^2 b.$$  \hfill (8)
Extrinsic distance and extrinsic balls

In order to understand the volume growth we need some previous concepts as the **extrinsic distance** and the **extrinsic balls**.

- The extrinsic distance is the restriction from the distance function in the ambient manifold to the submanifold.
- The extrinsic ball is the sublevel set defined by the extrinsic distance function.

**Definition of extrinsic distance**

Let $\varphi : M^n \to N$ be a complete, and proper immersion. Given two points $o, p \in M$, the extrinsic distance from $o$ to $p$ is

$$r_o(p) := \text{dist}^N(\varphi(o), \varphi(p))$$

where $\text{dist}^N$ denotes the geodesic distance in $N$. 

$\text{dist}^N$ denotes the geodesic distance in $N$. 

![Image of the definition of extrinsic distance]

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Definition of extrinsic ball

The extrinsic ball $D_R(o)$ of radius $R$ centered in $o \in M$ is the set of points whose extrinsic distance to $o$ is at most $R$

$$D_R(o) := \{ p \in M ; r_o(p) < R \}$$

Where $r_o(p)$ is the extrinsic distance from $o$ to $p$. 

$$D_R(o) := \{ p \in M ; r_o(p) < R \}$$ (10)
Volume growth I

With these extrinsic balls we can define the volume comparison quotient

$$Q_b(R) := \frac{\text{Vol}(D_R)}{\text{Vol}(B_{b}^{b,n})},$$

where $B_{R}^{b,n}$ stands for the geodesic ball of radius $R$ in $\mathbb{K}^n(b)$.

**Theorem volume growth (V. Palmer PLMS 1999)**

Let $\varphi : M \to N$ be a proper and minimal immersion into a Cartan-Hadamard ambient manifold $N$ ($K_N \leq b \leq 0$), then the volume comparison quotient $Q_b(R)$ is a non decreasing function on $R$. 
From the previous theorem we can define

**Definition**

Let \( \varphi : M \to N \) be a proper and minimal immersion into a Cartan-Hadamard ambient manifold \( N \) \((K_N \leq b \leq 0)\). \( M \) has **finite volume growth** if and only if

\[
\sup_R Q_b(R) = \lim_{R \to \infty} Q(R) < \infty.
\]

the volume comparison quotient has a finite upper bound.
Theorem, V Gimeno V. Palmer, JGEA 2013

Let $M^n \to \mathbb{H}^m(b)$ be a proper and complete minimal immersion $n > 2$. Suppose that

$$\|A\| \leq \frac{\delta(r)}{e^{2\sqrt{-br}}},$$

such that $\delta \to 0$ when $r \to \infty$.

Then :

1. $M$ has finite topological type.
2. $M$ has finite volume growth.
3. $\sup_R Q_b(R) \leq \mathcal{E}(M) = \text{ends of } M$. 
Let $M^2 \to \mathbb{H}^m(b)$ be a complete minimal immersion, suppose that

$$\int_{M^2} \| A \|^2 \, dV < \infty$$

then

$$\sup_R Q_b(R) \leq \frac{1}{4\pi} \int_{M^2} \| A \|^2 \, dV + \chi(M^2)$$
Let $M$ be a non-compact connected manifold. We define an equivalence relation in the set $A = \{ \alpha : [0, \infty) \to M | \alpha \text{ is a proper arc} \}$, by setting $\alpha_1 \sim \alpha_2$ if for every compact set $C \subset M$, $\alpha_1, \alpha_2$ lie eventually in the same component of $M - C$.

**Definition**

Each equivalence class in $E(M) = A / \sim$ is called an *end* of $M$. 
Given an exhaustion by compact sets \( \{K_i\} \) of the manifold \( P \) (\( K_i \subset K_{i+1} \) and \( \bigcup_{i \in \mathbb{N}} K_i = P \) ), the number of ends \( \mathcal{E}(P) \) of \( P \) is the supremum of the number of connected components with non compact closure of \( P - K_i \).

(see Tkachev’s paper Manuscripta Math. 82, 1994 and Anderson’s I.E.H.S. preprint 1984)
Some examples

1. The number of ends of any compact space is zero.
2. The real line $\mathbb{R}$ has two ends.
3. If $n > 1$, then the Euclidean space $\mathbb{R}^n$ has only one end. This is because $\mathbb{R}^n \setminus F$ has only one unbounded component for any compact set $F$.
4. The catenoid has two ends.

5. The periodic surface of Callahan-Hoffman-Meeks has infinitely many ends.
What I know? I

Theorem, V. Gimeno V. Palmer, PAMS 2013

Let $\varphi : M^n \to N$ be a proper complete minimal immersion in a Cartan-Hadamard ambient manifold $N$ ($K_N \leq b \leq 0$). Suppose that the submanifold has finite volume growth,

$$\sup_R Q(R) < \infty,$$

then

$$I_\infty(M) = (n - 1)\sqrt{-b}$$
Theorem, V. Gimeno, POTA 2013

Let \( \varphi : M^n \to N \) be a proper complete minimal immersion in a Cartan-Hadamard ambient manifold \( N (K_N \leq b \leq 0) \). Suppose that the submanifold has finite volume growth,

\[
\sup_R Q(R) < \infty,
\]

then

\[
\lambda^*(M) = \frac{-(n-1)^2 b}{4}
\]
Sketch of the proof I

For any $\Phi \in L^2_{1,0}(M) \setminus \{0\}$

$$\lambda^*(M) \leq \frac{\int_M |\nabla \Phi|^2 dV}{\int_M |\Phi|^2 dV}$$

Pick

$$\Phi : M \to \mathbb{R}; \quad \Phi = \phi_R \circ r.$$  

$$\phi_R(t) = \begin{cases} 
\sin\left(\frac{2\pi(t - \frac{R}{2})}{R}\right) & \text{if } t \in \left[\frac{R}{2}, R\right] \\
\frac{\text{Vol}(S^b_t)}{\text{Vol}(S^b_{\frac{R}{2}})^{\frac{1}{2}}} & \text{if } t \in \left[\frac{R}{2}, R\right] \\
0 & \text{otherwise.} 
\end{cases}$$
Sketch of the proof II

By the Rayleigh quotient definition and the coarea formula

\[ \lambda^*(M) \leq \frac{\int_M \langle \nabla \Phi, \nabla \Phi \rangle d\mu}{\int_M \Phi^2 d\mu} = \frac{\int_M (\phi')^2 \langle \nabla r_p, \nabla r_p \rangle d\mu}{\int_M \Phi^2 d\mu} \leq \frac{\int_M (\phi')^2 d\mu}{\int_M \Phi^2 d\mu} \]

\[\begin{align*}
\int_0^R \left[ \int_{\partial D_s} (\phi')^2 \frac{1}{|\nabla r|} \right] ds &= \int_0^R \left[ \int_{\partial D_s} \phi^2 \frac{1}{|\nabla r|} \right] ds \\
\int_0^R \left[ \int_{\partial D_s} \phi^2 \frac{1}{|\nabla r|} \right] ds &= \int_{\frac{R}{2}}^R \phi^2(s) \left[ \int_{\partial D_s} \frac{1}{|\nabla r|} \right] ds \tag{12}
\end{align*}\]

\[\begin{align*}
\int_{\frac{R}{2}}^R (\phi'(s))^2 (\text{Vol}(D_s))' ds &= \int_{\frac{R}{2}}^R \phi^2(s) (\text{Vol}(D_s))' ds \\
&= \int_{\frac{R}{2}}^R (\phi'(s))^2 (\text{Vol}(D_s))' ds
\end{align*}\]

From the definition of \( Q_b \) and taking into account that \( Q \) is a non-decreasing function

\[ (\ln Q_b(s))' = \frac{(\text{Vol } D_s)'}{\text{Vol } D_s} - \frac{\text{Vol}(S^b_s)}{\text{Vol}(B^b_s)} \geq 0 \tag{13} \]

\[\begin{align*}
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\end{align*}\]
Sketch of the proof III

So,

\[ Q_b(s) \text{Vol}(S_s^b) \leq (\text{Vol}(D_s))' \leq (\ln Q_b(s))' \text{Vol}(B_s^b)Q_b(s) + Q_b(s) \text{Vol}(S_s^b) \]  

(14)

Lemma

There exists an upper bound function \( \Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) to

\[ \frac{\int_0^R (\phi')^2 \text{Vol}(S_s^b) ds}{\int_0^R \phi^2 \text{Vol}(S_s^b) ds} \leq \Lambda(R) \]  

(15)

such that

\[ \lim_{R \rightarrow \infty} \Lambda(R) = \frac{-(n - 1)^2 b}{4} \]  

(16)
Denoting now,

\[
F(R) := \left( \frac{(m-1)^2}{4} \cot_b(R/2)^2 + \frac{4\pi^2}{R^2} + \frac{2(m-1)\pi}{R} \cot_b(R/2) \right)
\]

\[
\delta(R) := \int_{\frac{R}{2}}^{R} (\ln Q(s))' \, ds,
\]

\[
\lambda^*(M) \leq \frac{Q(R)}{Q(\frac{R}{2})} \left[ \frac{\text{Vol}(B^b_R)}{\text{Vol}(S^b_R)} \frac{4}{R} F(R) \delta(R) + \Lambda(R) \right]
\]

Letting \( R \) tend to infinity and taking into account that

\[
\lim_{R \to \infty} F(R) = -\frac{(n-1)^2 b}{4},
\]

\[
\lim_{R \to \infty} \delta(R) = 0,
\]

\[
\lim_{R \to \infty} \frac{\text{Vol}(B^b_R)}{\text{Vol}(S^b_R)} \frac{4}{R} = \begin{cases} \frac{4}{m-1} & \text{if } b = 0, \\ 0 & \text{if } b < 0. \end{cases}
\]

\[
\lim_{R \to \infty} \frac{Q(R)}{Q(\frac{R}{2})} = 1.
\]
An improvement? I

Theorem, S Ilias, B. Nelli, M. Soret, Arxiv aug 2013

Let $\varphi : M^n \to N$, $N$ Cartan-Hadamard, if

$$\sup_{R} Q_b(R) < \infty$$

then:

- $I_\infty(M) \leq (m - 1)\sqrt{-b}$
- if $M$ is minimal, $\lambda^*(M) = \frac{-(m-1)^2b}{4}$

They make use of the volume entropy $\mu_M$ of $M$

$$\mu_M := \limsup_{R \to \infty} \left( \frac{\ln(\text{Vol}(D_R))}{R} \right) < \infty.$$ 

Since

$$\text{Vol}(D_R) \leq \sup_{R} Q_b(R) \text{Vol}(B^b_R)$$
An improvement? II

\[ \mu_M = \limsup_{R \to \infty} \left( \frac{\ln(\sup R Q_b(R))}{R} + \frac{\ln(\text{Vol}(B^b_R))}{R} \right) < \infty. \]

Therefore,

\[ \mu_M := \mu_{\mathbb{H}^n}(b). \]

Independence on the volume growth

\[ \sup_{R} Q_b(R) ? \]

We only need its finiteness.
We have seen

$$\sup_{R} Q_{b}(R) \sim E(M)$$

There exists an other relation?
Outline

1. Cheeger isoperimetric constant and fundamental tone

2. Capacity
   - Volume growth and number of ends
Given a compact set $K \subset M$ in a Riemannian manifold $M$ and an open set $\Omega \subset M$ containing $K$, we call the couple $(K, \Omega)$ a capacitor. Each capacitor has capacity defined by

$$\text{Cap}(K, \Omega) := \inf_u \int_{\Omega \setminus K} \| \nabla u \| \, d\mu,$$

where the inf is taken over all Lipschitz functions $u$ with compact support in $\Omega$ such that $u = 1$ on $K$.

When $\Omega$ is precompact, the infimum is attained for the function $u = \Psi$ which is the solution of the following Dirichlet problem in $\Omega \setminus K$:

$$\begin{cases}
\Delta \Psi = 0 \\
\Psi|_{\partial K} = 0 \\
\Psi|_{\partial \Omega} = 1
\end{cases}$$

(20)
From a physical point of view, the capacity of the capacitor \((K, \Omega)\) represents the total electric charge (generated by the electrostatic potential \(\Psi\)) flowing into the domain \(\Omega \setminus K\) through the interior boundary \(\partial K\). Since the total current stems from a potential difference of 1 between \(\partial K\) and \(\partial \Omega\), we get from Ohm’s Law that the effective resistance of the domain \(\Omega \setminus K\) is

\[
R_{\text{eff}}(\Omega \setminus K) = \frac{1}{\text{Cap}(K, \Omega)}.
\] (21)
La ecuación se ilustra gráficamente, mostrando la representación de un potencial electrostático y las corrientes totales fluyendo a través de la frontera de un dominio.

La ecuación se refiere a la representación del potencial electrostático generado por una concentración de carga en el interior de un dominio $K$.

Además, la cantidad $\text{Cap}(K, \Omega)$ representa la corriente total fluyendo en el interior de $\Omega$ a través de $\partial K$. 

El diagrama muestra la distribución de campo eléctrico en un sistema de batería, con flechas indicando la dirección de la corriente y la frontera del dominio $\partial K$. 

La batería está etiquetada como "BATERÍA" en el diagrama.
Capacity of extrinsic annuli

Given an isometric immersion $\varphi : M \to N$, the extrinsic annulus is

$$A_{\rho,R} := \{ x \in M \mid \rho \leq r(x) \leq R \}$$

**Theorem, S. Markvorsen V. Palmer, GAFA 2002**

Let $\varphi : M^n \to N$ be a proper and minimal immersion into a Cartan-Hadamard ambient manifold with curvatures bounded from above by $K_n \leq b \leq 0$, then

$$\text{Cap}(A_{\rho,R}) \geq \text{Cap}(A_{\rho,R}^{K_n b}).$$
Let $\varphi : M^n \to N$ be a proper and minimal immersion into a Cartan-Hadamard ambient manifold with curvatures bounded from above by $K_n \leq b \leq 0$, then

$$1 \leq \frac{\text{Cap}(A_{\rho,R})}{\text{Cap}(A_{K_n b,\rho,R})} \leq \sup_R Q_b(R).$$
Catenoid
Theorem, Jorge-Meeks

Let $M^2$ be a minimal surface embedded in $\mathbb{R}^3$ with finite total curvature, then

$$\sup_R Q(R) = \mathcal{E}(M^2) = \text{number of ends of } M.$$
Capacity and number of ends

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Capacity and number of ends
Scherk’s singly periodic surface
The Scherk’s singly periodic surface has

$$\sup_R Q(R) = 2.$$
Volume growth number of ends I

Theorem, Anderson + Qing Chen, Manuscripta Math., 1997

Let $M$ be an $n$–dimensional complete properly immersed minimal submanifold in $\mathbb{R}^m$ which satisfies

$$\limsup R \frac{\|A\|}{r} = 0$$

Then

$$\lim_{R \to \infty} \frac{\text{Vol}(D_R)}{\omega_n R^n} = \mathcal{E}(M) < \infty.$$
Model space

A $w$–model space $M^n_w$ is a simply connected $n$-dimensional smooth manifold $M^n_w$ with a point $o_w \in M^n_w$ called the center point of the model space such that $M^n_w - \{o_w\}$ is isometric to a smooth warped product with base $B^1 = (0, \Lambda) \subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{n-1} = S^{n-1}_1$ (i.e. the unit $(n-1)$–sphere with standard metric), and positive warping function $w : [0, \Lambda) \to \mathbb{R}_+$. Namely:

$$g_{M^n_w} = \pi^* (g_{(0,\Lambda)}) + (w \circ \pi)^2 \sigma^* (g_{S^{n-1}_1}), \quad (22)$$

being $\pi : M^n_w \to (0, \Lambda)$ and $\sigma : M^n_w \to S^{n-1}_1$ the projections onto the factors of the warped product.
Volume growth number of ends III

Examples

\[ K^n_b = M^n_{wb}. \]

\[ w_b(r) = \begin{cases} 
\frac{1}{\sqrt{|b|}} \sin(\sqrt{b}r) & \text{if } b > 0 \\
1 & \text{if } b = 0 \\
\frac{1}{\sqrt{-b}} \sinh(\sqrt{-b}r) & \text{if } b < 0
\end{cases} \]

Balanced models

Balanced from below:

\[ \frac{\text{Vol}(B^w_r)}{\text{Vol}(S^w_r)} \frac{w'(r)}{w(r)} \geq \frac{1}{m} \]

Balanced from above:

\[ \frac{\text{Vol}(B^w_r)}{\text{Vol}(S^w_r)} \frac{w'(r)}{w(r)} \leq \frac{1}{m - 1} \]
Theorem, V. Gimeno, V. Palmer, JGEA 2013

Let $\varphi : M^n \to M^m_w$ be a proper and complete minimal immersion into a balanced from below model space $M^m_w$. Suppose that:

- $n > 2$,
- $w'(r) \geq d > 0$.
- $w'(r)w(r)\|A\| \leq \epsilon(r)$ such that $\epsilon \to 0$ when $r \to \infty$.

Then, $M$ has finite topological type and

$$1 \leq \lim_{R \to \infty} \frac{\text{Vol}(D_R)}{\text{Vol}(B^w_R)} \leq \mathcal{E}(M).$$
Theorem, Qing Chen, Manuscripta Math., 1997

Let $M^n$ be a complete, proper and $n$–dimensional minimal submanifold of $\mathbb{R}^m$. Suppose that:

$$\sup_{R > 0} \frac{\text{Vol}(D_R)}{\omega_n R^n} < \infty.$$ 

Then

$$\mathcal{E}(M) \leq \sup_{R > 0} \frac{\text{Vol}(D_R)}{\omega_n R^n}.$$
Volume growth number of ends VI

Theorem, V. Gimeno and S. Markvorsen, in preparation

Let $\varphi : M^n \to N^m$ be a proper minimal and complete immersion. Where:

- $N$ possesses a pole
- The sectional curvatures $K_N$ of $N$ are bounded by the radial curvatures $K_w$ of a balanced from below model space $M^n_w$

$$K_N(p) \leq K_{M^n_w}(r(p)) = -\frac{w''}{w}(r(p)).$$

- $w' > 0$ and there exist $R_0$ such that $K_{M^n_w}(R) \leq 0$ for any $R > R_0$

$$\limsup_{t \to \infty} \left( \int_0^t \frac{w(s)^{m-1}ds}{t^m/m} \right) = C_w < \infty.$$

Then, if $M$ has finite $w$–volume growth,

$$\mathcal{E}(P) \leq 2^m C_w \lim_{t \to \infty} \frac{\text{Vol}(D_t)}{\text{Vol}(B^w_t)}. \quad (23)$$
Thanks!!