LAGRANGIAN SUBMANIFOLDS REALISING EQUALITY IN A BASIC INEQUALITY

LUC VRANCKEN

Abstract

In the study of geometry of submanifolds one is mainly interested in the relation between the intrinsic properties of the submanifold (i.e. properties which depend only on the submanifold itself) and extrinsic properties (i.e. properties which depend on the immersion of the submanifold in the surrounding space).

The main intrinsic invariants are those which can be constructed using the metric (Levi Civita connection, curvature tensor). For surfaces in Euclidean 3-space one has the following relation between the Gaussian curvature K and the length of the mean curvature vector H, i.e.

$$K \leq H^2,$$

with equality everywhere if and the surface is an open part of a plane or a sphere.

An example of an intrinsic invariant in higher dimensions is the δ-invariant introduced by B. Y. Chen in the study of submanifolds of real space forms. It is defined by

$$\delta_M(p) = \tau(p) - inf_{\pi \subset T_pM} K(\pi),$$

where

(1) $K(\pi)$ is the sectional curvature of the plane π
(2) $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$

In this lecture we are interested in Lagrangian submanifolds M^n into the complex projectif space $\mathbb{C}P^n(4)$. Note that a submanifold is called Lagrangian if the complex structure J of the ambient space interchanges the tangent and the normal space. For such submanifolds, it is possible to derive an inequality between Chen's invariant (which is an intrinsic invariant) and the length of the mean curvature vector (which is the main extrinsic invariant). Namely,

$$\delta_M \leq \tfrac{1}{2}(n+1)(n-2) + \tfrac{n^2(2n-3)}{2(2n+3)} H^2.$$

We will give a survey of the results related to the above inequality. We will in particular try to understand the submanifolds which at every point realise equality in the above inequality. Surprisingly this class is very large and seems in some special cases related to various classes of minimal surfaces. Here the cases $H = 0$ and $H \neq 0$ need to be treated separately and lead to completely different results.

In particular if $H \neq 0$ it is possible to show that the dimension needs to be 3 and in that dimension a complete classification can be obtained. Remark that it is also possible to prove that complete submanifolds which at every point realise equality can only occur in dimension 3. In the minimal case, we present a geometric construction in dimension 3. Unfortunately this construction is only valid in this dimension. We finish by showing that in the minimal case many examples exist in every dimension (even though if the dimension is greater then 3 they are not complete) and we indicate some directions of future research.

Most of the results presented during this lecture are joint work with J. Bolton. Other results were obtained also in collaboration with F. Dillen.