Singularities of the hyperbolic Schwarz map for the hypergeometric differential equation

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The Schwarz map of the hypergeometric differential equation is studied since the beginning of the last century. Its target is the complex projective line, the 2-sphere. In this talk, I will discuss the *hyperbolic Schwarz map*, whose target is the hyperbolic 3-space, and which can be considered as a lifting of the Schwarz map to the 3-space.

The hypergeometric differential equation is defined as

$$x(1-x)u'' + \{c-(a+b+1)x\}u' - abu = 0$$

and the Schwarz map as

$$s : X = \mathbf{C} - \{0,1\} \ni x \longmapsto u_0(x) : u_1(x) \in Z \cong \mathbf{P}^1,$$

where u_0 and u_1 are linearly independent solutions of the equation and $\mathbf{P}^1$ stands for the complex projective line. The map $x \mapsto u_0'(x) : u_1'(x)$ is called the derived Schwarz map.

We propose a variation of the Schwarz map as follows: Change the equation into the so-called SL-form:

$$u'' - q(x)u = 0,$$

and transform it to the matrix equation

$$\frac{d}{dx}(u,u') = (u,u')\Omega, \quad \Omega = \begin{pmatrix} 0 & q(x) \\ 1 & 0 \end{pmatrix}.$$

The *hyperbolic Schwarz map* is defined to be the composition of the (multi-valued) map

$$X \ni x \longmapsto H = U(x)\,{}^t\overline{U}(x) \in \mathrm{Her}^+(2)$$

and the natural projection $\mathrm{Her}^+(2) \to \mathbf{H}^3 := \mathrm{Her}^+(2)/\mathbf{R}^+$, where $U(x)$ is a fundamental solution of the system, $\mathrm{Her}^+(2)$ the space of positive-definite Hermitian matrices of size 2, and $\mathbf{R}^+$ the multiplicative group of positive real numbers; the space $\mathbf{H}^3$ is the *hyperbolic* 3-*space*. Note that the target of the hyperbolic Schwarz map is $\mathbf{H}^3$, whose boundary is $\mathbf{P}^1$, which is the target of the Schwarz map. Note that the monodromy group of the system acts naturally on $\mathbf{H}^3$.

The image surface is one of the *flat fronts* in $\mathbf{H}^3$, which is a flat surface with a certain kind of singularities. Moreover, the classical Schwarz map s is recovered as the *hyperbolic Gauss map* of the hyperbolic Schwarz map as a flat front. The works by J. A. Gálvez, A. Martínez and F. Milán [2000, Math. Ann.], and M. Kokubu, M. Umehara, and K. Yamada [2004, Pacific J. Math.] of constructing flat surfaces in the three-dimesnional hyperbolic space are fundamental to our study. Since any closed nonsigular flat surface is isometric to a horosphere or a hyperbolic cylinder, such surfaces have necessarily singularities: generic singularities of flat fronts are cuspidal edges and swallowtail singularities.

We treat the following subjects:

- To visualize the image of the hyperbolic Schwarz map when the monodromy group is a finite group or a typical Fuchsian group and to make clear where the sigularities appear on the image,
- To list all kinds of singularities of the map for the confluent hypergeometric equation and to compute the positions of singularities, especially swallowtail singularities,
- To know of the asymptotic behavior of the map for the Airy differential equation and for differential equations of the second order with irregular singularity at infinity, and
- To give a geometric description of the relation of the hyperbolic Schwarz map with the Schwarz map, and also with the derived Schwarz map.

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