Poincaré Duality Angles on Riemannian Manifolds with Boundary

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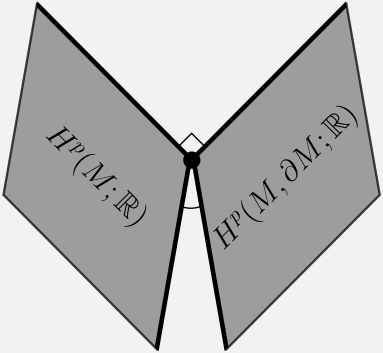
Realizing cohomology groups as spaces of differential forms

Let M^n be a compact Riemannian manifold with non-empty boundary ∂M.



de Rham's Theorem

Suppose M^n is a compact, oriented, smooth manifold. Then

$$H^p(M;\mathbb{R}) \cong \mathcal{C}^p(M)/\mathcal{E}^p(M),$$

where $\mathcal{C}^p(M)$ is the space of closed p-forms on M and $\mathcal{E}^p(M)$ is the space of exact p-forms.

If M is Riemannian, the metric induces an L^2 inner product on $\Omega^p(M)$:

$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$

When M is closed, the orthogonal complement of $\mathcal{E}^p(M)$ inside $\mathcal{C}^p(M)$ is

$$\mathcal{H}^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0\}$$

Kodaira called this the space of *harmonic p-fields* on M.

Hodge's Theorem

If M^n is a closed, oriented, smooth Riemannian manifold,

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$

Hodge–Morrey–Friedrichs Decomposition

Define $i : \partial M \hookrightarrow M$ to be the natural inclusion.

The L^2-orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}^p_N(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^* \star \omega = 0\}.$$

Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p_N(M).$$

Hodge–Morrey–Friedrichs Decomposition (continued)

The relative cohomology appears as

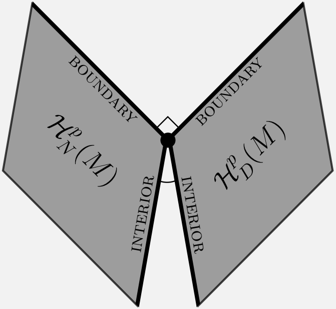
$$H^p(M, \partial M; \mathbb{R}) \cong \mathcal{H}_D^p(M).$$

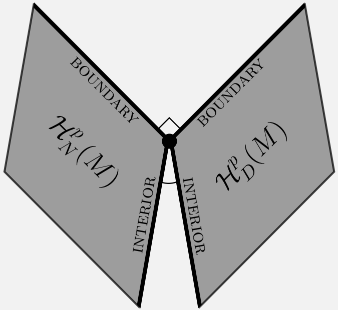
$$\mathcal{H}_D^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^*\omega = 0\}.$$

The concrete realizations of $H^p(M;\mathbb{R})$ and $H^p(M,\partial M;\mathbb{R})$ meet only at the origin:

$$\mathcal{H}^p_N(M) \cap \mathcal{H}^p_D(M) = \{0\}$$

...but they are not orthogonal!





Definition (DeTurck–Gluck)

The *Poincaré duality angles* of the Riemannian manifold M are the principal angles between the interior subspaces.

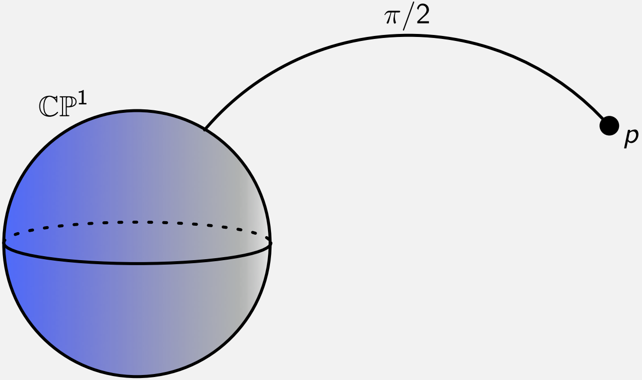
What do the Poincaré duality angles tell you?

Guess

If M is "almost" closed, the Poincaré duality angles of M should be small.

Consider $\mathbb{CP}^2$ with its usual Fubini-Study metric. Let $p \in \mathbb{CP}^2$. Then define

$$M_r := \mathbb{CP}^2 - B_r(p).$$



Topology of M_r

∂M_r is a 3-sphere.

M_r is the D^2-bundle over $\mathbb{CP}^1$ ($\simeq S^2(1/2)$) with Euler characteristic 1.

M_r has absolute cohomology in dimensions 0 and 2.

M_r has relative cohomology in dimensions 2 and 4.

Therefore, M_r has a single Poincaré duality angle θ_r between $\mathcal{H}^2_N(M_r)$ and $\mathcal{H}^2_D(M_r)$.

Find harmonic 2-fields

So the goal is to find closed and co-closed 2-forms on M_r which satisfy Neumann and Dirichlet boundary conditions.

Such 2-forms must be isometry-invariant.

$$\mathrm{Isom}_0(M_r) = SU(2).$$

Find closed and co-closed $SU(2)$-invariant forms on M_r satisfying Neumann and Dirichlet boundary conditions

The Poincaré duality angle for M_r

$$\cos\theta_r = \frac{1 - \sin^4 r}{1 + \sin^4 r}.$$

As $r \to 0$, the Poincaré duality angle $\theta_r \to 0$.

As $r \to \pi/2$, $\theta_r \to \pi/2$.

Generalizes to $\mathbb{CP}^n - B_r(p)$.

Consider

$$N_r := G_2\mathbb{R}^n - \nu_r(G_1\mathbb{R}^{n-1}).$$

Theorem

- *As $r \to 0$, all the Poincaré duality angles of N_r go to zero.*
- *As r approaches its maximum value of $\pi/2$, all the Poincaré duality angles of N_r go to $\pi/2$.*

Conjecture

If M^n is a closed Riemannian manifold and N^k is a closed submanifold of codimension ≥ 2, the Poincaré duality angles of

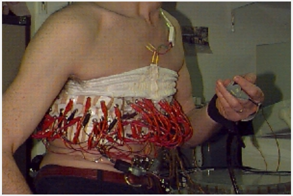
$$M - \nu_r(N)$$

go to zero as $r \to 0$.

A question

What can you learn about the topology of M from knowledge of ∂M?

Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.



Suppose f is a potential on the boundary of a region $M \subset \mathbb{R}^3$.

Then f extends to a potential u on M, where

$$\Delta u = 0, \quad u|_{\partial M} = f.$$

If γ is the conductivity on M, the current flux through ∂M is given by

$$(\gamma \nabla u) \cdot \nu = \gamma \frac{\partial u}{\partial \nu}$$

The Dirichlet-to-Neumann map

The map $\Lambda_{\mathrm{cl}} : C^\infty(\partial M) \to C^\infty(\partial M)$ defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

is the classical *Dirichlet-to-Neumann map*.

Theorem (Lee-Uhlmann)

If M^n is a compact, analytic Riemannian manifold with boundary, then M is determined up to isometry by Λ_{cl}.

Generalization to differential forms

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$$

Theorem (Belishev–Sharafutdinov)

The data $(\partial M, \Lambda)$ completely determines the cohomology groups of M.

Connection to Poincaré duality angles

Define the *Hilbert transform* $T := d\Lambda^{-1}$.

Theorem

If $\theta_1, \ldots, \theta_k$ *are the Poincaré duality angles of* M *in dimension* p*, then the quantities*

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of T^2*.*

Belishev and Sharafutdinov posed the following question:

> *Can the multiplicative structure of cohomologies be recovered from our data $(\partial M, \Lambda)$? Till now, the authors cannot answer the question.*

Theorem

The mixed cup product

$$\cup : H^p(M;\mathbb{R}) \times H^q(M,\partial M;\mathbb{R}) \to H^{p+q}(M,\partial M;\mathbb{R})$$

is completely determined by the data $(\partial M, \Lambda)$ when the relative class is restricted to come from the boundary subspace.

Some questions

- Poincaré duality angles for $G_4\mathbb{R}^8 - \nu_r(G_3\mathbb{R}^7)$? Other "Grassmann manifolds with boundary"?
- What is the limiting behavior of the Poincaré duality angles as the manifold "closes up"?
- Can the full mixed cup product be recovered from $(\partial M, \Lambda)$? What about other cup products?
- Can the L^2 inner product on $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$ be recovered from $(\partial M, \Lambda)$?

Thanks!