Casey Douglas - Research Statement

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1. Introduction

My current research is in the theory of minimal surfaces. Via classical and geometric techniques, I have established the existence of a new family of doubly periodic, minimal tori that limit on the singly periodic, genus-one helicoid. This involved solving a two dimensional period problem, which was accomplished by perturbing a one dimensional period problem. To produce the limiting helicoidal surface, geometric arguments were used to restrict possible degenerations of the associated minimal surfaces.

2. Background

Minimal Surfaces. There are a number of ways to define a minimal surface, (see, for example, [5] or [10]), but one of the more common or useful definitions involves **Weierstrass data**. A result of Osserman tells us that every finite total curvature minimal surface is conformally a compact Riemann surface with finitely many punctures [15]. The map $X : \mathcal{R} \to \mathbb{R}^3$ that parameterizes our punctured Riemann surface, $\mathcal{R}$, as a minimal surface admits an integral representation given by

$$X(z) = \mathrm{Re} \int_{\cdot}^{z} \left(\frac{1}{2}\left(\frac{1}{g} - g\right) dh, \frac{i}{2}\left(\frac{1}{g} + g\right) dh, dh \right)$$

where g is a holomorphic function, dh is a holomorphic 1-form, and z is a local coordinate on the surface $\mathcal{R}$. The pair (g, dh) is referred to as the **Weierstrass data** for the minimal surface.

Both g and dh have geometric significance. As the notation suggests, dh is the differential of the height function, and g is the Gauss map composed with stereographic projection. To construct a desired minimal surface, it suffices to determine appropriate g and dh. In order for the surface to be unbranched, one first has to balance the zeroes and poles of g against the zeroes of dh. In order for the map X to be well defined, one has to solve the period problem(s)

$$\mathrm{Re}\left(\int_\gamma \frac{1}{2}\left(\frac{1}{g} - g\right) dh \right) = 0$$

$$\mathrm{Re}\left(\int_\gamma \frac{i}{2}\left(\frac{1}{g} + g\right) dh \right) = 0$$

$$\mathrm{Re}\left(\int_\gamma dh \right) = 0$$

where the integrals are taken over all generators γ of $H_1(\mathcal{R}; \mathbb{C})$. The first two equations are often referred to as the **horizontal period problem**, while the last is the **vertical period problem**. The horizontal period problem can be rewritten as a single complex equation

$$\int_\gamma g\,dh = \overline{\int_\gamma \frac{1}{g} dh}$$

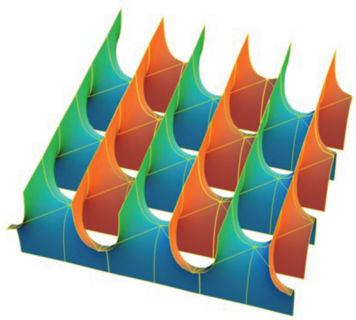
again for all γ that generate $H_1(\mathcal{R}; \mathbb{C})$. If $\mathcal{R}$ has genus k and n punctures, there are $3(2k + n - 1)$ real conditions to satisfy. Moreover, if $\mathcal{R}$ has high genus, then the function g and 1-form dh can be difficult to determine. In summary, topologically complicated minimal surfaces are often difficult to construct via Weierstrass data.

When $\mathcal{R}$ is a punctured sphere, the period problem typically reduces to a condition on the residues of $gdh, (1/g)dh$ and dh, namely that they are purely real. A good example is given by Scherk's doubly periodic surface, which is defined on $\hat{\mathbb{C}} - \{\pm e^{\pm i\theta}\}$ by the data

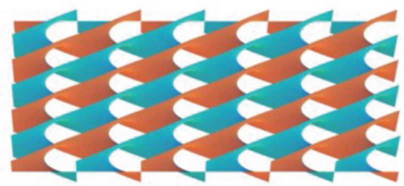
$$g(z) = z$$

$$dh = \frac{izdz}{\prod(z \pm e^{\pm i\theta})}$$

Only the vertical period problem is solved for this data, producing a doubly periodic surface in $\mathbb{R}^3$ that is defined over the lattice $(\sec\theta, \csc\theta, 0)$, as pictured below.

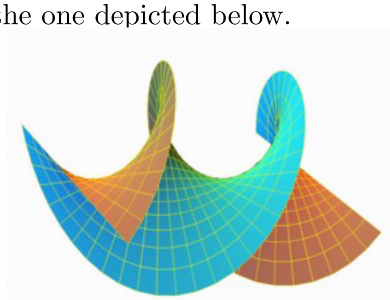


$\theta = \pi/4$

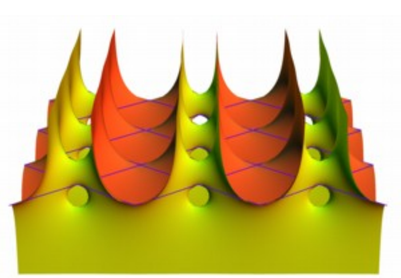


$\theta < \pi/4$

If we let $\theta \to 0$, these surfaces converge to a singly periodic helicoid whose axis of revolution is contained in the (x, y) plane, as the one depicted below.

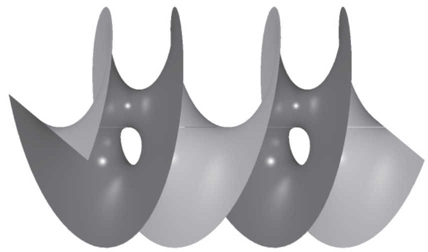


Karcher's Surfaces and the Singly Periodic Genus-One Helicoid. By imposing symmetry on a desired minimal surface, the number of period problems one has to solve is greatly reduced. Karcher succeeded in adding a handle to genus-0 surfaces (see [11, 12, 13] and the image below) by making use of imposed symmetry and appealing to basic elliptic function theory. In particular, he added a handle to Scherk's doubly periodic surface for $2\theta = \pi/2$ under the assumption that the underlying torus was square; this reduced what should have been a two dimensional period problem to a one dimensional problem.



Scherk-Karcher Surface

Motivated by Karcher's construction, Hoffman-Karcher-Wei [9] conjectured that a handle could be added to Scherk's surfaces for all values of θ. Moreover, they produced numerical estimates suggesting that as $\theta \to 0$ the surfaces limit on a singly periodic, genus-one helicoid (pictured below). They succeeded in establishing the existence of this proposed limit surface; in fact, they showed that the surface is unique and embedded. However, the existence of genus-1 Scherk surfaces for arbitrary θ was not addressed.



Handle Addition, Flat Structures, and Extremal Length. There are other ways to add handles to minimal surfaces. One such method was developed by Weber-Wolf [20], whereby **flat structures** for the proposed 1-forms gdh and $(1/g)dh$ are determined up to a number of free parameters, with the horizontal period conditions cutting down this number to a typically manageable size.

Specifically, given any 1-form α on a Riemann surface $\mathcal{R}$, one can define a line element ds_α by

$$ds_\alpha = |\alpha| = |f(z)||dz|$$

where z is a local coordinate on $\mathcal{R}$ and $\alpha = f(z)dz$. Because $f(z)$ is meromorphic, away from the zeroes and poles of α the metric ds_α is flat. Near a zero or pole of order k, it is isomorphic to a Euclidean cone metric with cone angle $2\pi(k+1)$.

The developing map

$$z \mapsto \int_{\cdot}^{z} \alpha$$

is conformal and takes ds_α-geodesics to Euclidean lines. Often times, one can develop the entire surface $\mathcal{R}$ as a polygonal region in $\hat{\mathbb{C}}$ with various edges identified. These regions are called the **flat structure representations** of α.

For a minimal surface, the gdh and $(1/g)dh$ flat structures will enjoy a special relationship. The horizontal period problem requires homologous ds_{gdh}-geodesics and $ds_{(1/g)dh}$-geodesics to develop into conjugate line segments. Because the cone metrics ds_{gdh} and $ds_{(1/g)dh}$ are non-positively curved, geodesics in a particular homology class are guaranteed to exist (see [4] or [18]), and lines of symmetry are always geodesics for both metrics (in fact, for any ds_α metric).

Conversely, one can construct a desired minimal surface by first determining necessary gdh and $(1/g)dh$ flat structures. The solution to the horizontal period problem is assumed, built into the structures, but various edge-lengths remain undetermined. The question then becomes one of conformal type, for the 1-forms gdh and $(1/g)dh$ will be defined on the same Riemann surface $\mathcal{R}$ if and only if their associated flat structures are conformally equivalent.

One is then required to show that for some choice of the undetermined parameters, enough conformal invariants agree, ensuring that the gdh and $(1/g)dh$ flat structures define the same, underlying Riemann surface. Using the relationship

$$dh = \sqrt{(gdh)\left(\frac{1}{g}dh\right)}$$

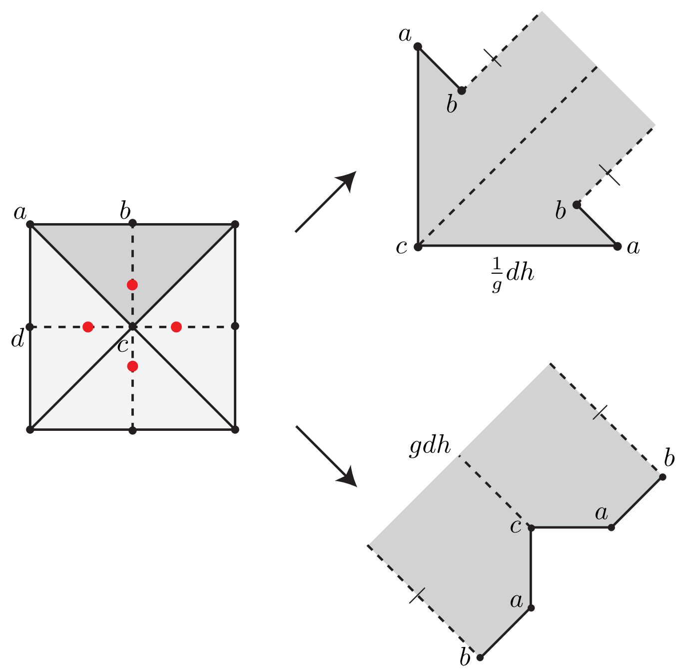
the vertical period problem is typically easy to solve.

The **Extremal Length** of a set of curves is often among the more useful conformal invariants for these purposes. Given a set of rectifiable curves Γ, and denoting the set of all Borel-measurable, conformal metrics by $\mathcal{M} = \{\rho \geq 0\}$, the Extremal Length is given by

$$\mathrm{Ext}_{\mathcal{R}}(\Gamma) = \sup_{\rho \in \mathcal{M}} \frac{\inf_{\gamma \in \Gamma} (L_\rho(\gamma))^2}{A_\rho(\mathcal{R})}$$

where $L_\rho(\gamma)$ denotes the ρ-length of a curve γ, and $A_\rho(\mathcal{R})$ denotes the ρ-area of $\mathcal{R}$. Using the gdh and $(1/g)dh$ flat structures to estimate various extremal lengths, by means of the Intermediate Value Theorem or via the properness of an associated function, one is able to argue that conformally equivalent flat structures exist. For more details on the notion of extremal length see [1] or [2].

Flat Structures for Karcher's Surface. When the surface $\mathcal{R}$ has sufficient symmetry, the conformal map relating the two flat structures is required to be edge preserving, too. This happens, for example, with Karcher's genus-one version of Scherk's doubly periodic surface with $\theta = \pi/4$:

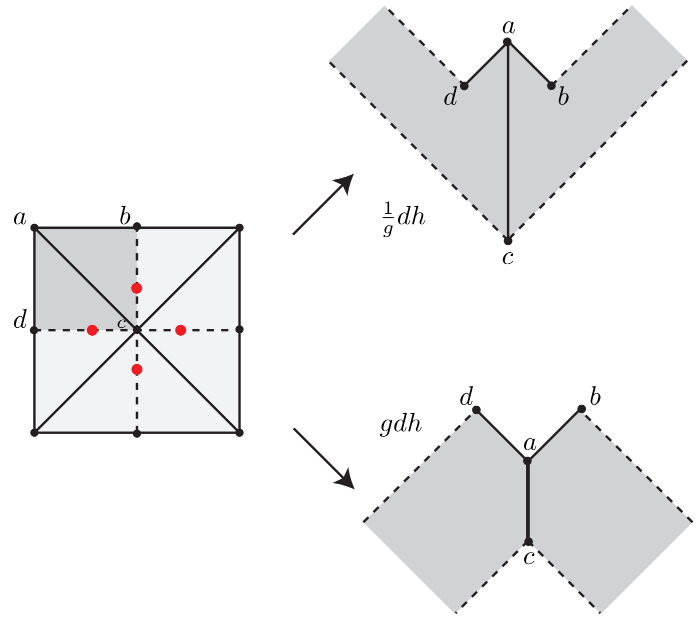


The picture above represents quarters of the full gdh and $(1/g)dh$ flat structures. Points that share a label are identified, and the line segments joining b to infinity are also identified, as indicated by the slash marks. A conformal map between the two domains necessarily exists, and it needs to take edges joining labeled points in one domain to edges joining similarly labeled points in the other domain.

In fact, as proven in [21], more general genus-k versions of this surface exist and are constructed so as to enjoy maximal symmetry. The flat structures for these higher genus surfaces also enjoy edge correspondence.

The large amount of symmetry present in these and other cases allows one to work with simply-connected domains bordered by lines meeting at angles of $\pi/2$ and $3\pi/2$. Such domains are called **orthodisks** (see [21]), and they significantly facilitate the approximation of various extremal lengths and their dependence on the flat structures' free parameters.

The previously depicted gdh and $(1/g)dh$ flat structures for the Scherk-Karcher surface are *not* orthodisks. However, orthodisks can be obtained by developing the quarters indicated below



The only undetermined parameter in the structures above is ℓ, which is the length of the segment joining a and d (or, equivalently, the segment joining a and b). Using an appropriate family of curves Γ, it can be shown that there exists precisely one value of $\ell \in (0, \infty)$ so that $\mathrm{Ext}_{gdh}(\Gamma) = \mathrm{Ext}_{\frac{1}{g}dh}(\Gamma)$, where $\mathrm{Ext}_{\alpha}(\Gamma)$ denotes the extremal length of Γ as computed on the flat structure for α. Hence, there exists a unique choice of ℓ so that the two domains are conformally equivalent.

3. Current Work

Let $S(k, 2\theta)$ denote a putative, complete, genus-k, doubly periodic minimal surface in $\mathbb{R}^3$ with four vertical, annular ends meeting at angles 2θ and $\pi - 2\theta$, where $\theta \in (0, \pi/2)$; we refer to such a surface as a perturbed, genus-k Scherk surface. Let $\mathcal{H}(k)$ denote a putative, singly periodic, genus-k helicoid. The fact that

$$\lim_{\theta \to 0} S(0, 2\theta) = \mathcal{H}(0)$$

suggests that an analogous result should hold for arbitrary k. That is, one conjectures

$$\lim_{\theta \to 0} S(k, 2\theta) = \mathcal{H}(k)$$

Although the existence of $\mathcal{H}(1)$ has been established for some time [9], the existence of $S(1, 2\theta)$ for arbitrary θ was only established recently. Using the method of the support function, Baginski-Ramos Batista [3] claimed the existence and conjectured limiting behavior of the surfaces $S(1, 2\theta)$. Using standard perturbation techniques and the flat structure techniques outlined above, I have also obtained these results.

Uniqueness and Perturbations of the Initial Surface. Unfortunately, because the surfaces $S(1, 2\theta)$ do not, in general, possess as much symmetry as the surface $S(1, \pi/2)$ does, using the gdh and $(1/g)dh$ flat structures to estimate various extremal lengths presents a new challenge; orthodisks are not available in this setting, as the flat structures necessarily have non-trivial topology, and so a different or modified approach is needed.

Indeed, flat structures with topology are not as amenable as simply connected ones, such as orthodisks. For the latter, extremal lengths provide coordinates for Teichmüller Space, and estimates are available via Schwarz-Christoffel mappings. To circumvent the obstructions posed by topologically complex flat structures, I first improved the existence results of $S(1, \pi/2)$, relaxing a symmetry requirement and establishing uniqueness. Specifically, I proved the following

Theorem 1. *Let $\mathcal{R} = \mathbb{C}/\{1, i\} - \{a_1, a_2, a_3, a_4\}$ denote a four-times punctured, square torus, where the punctures are placed with respect to the lattice's rhombic symmetry. Then, up to a re-ordering of the a_i, and a shift and rotation of the torus, there is precisely one way to place the punctures so that $\mathcal{R}$ embeds into $\mathbb{R}^3$ as $S(1, 2\theta)$. Moreover, the angle between the ends is necessarily given by $2\theta = \pi/2$.*

In [12] and [21], the punctures a_i were assumed to be placed with respect to both the torus' rectangular and rhombic lines of symmetry. Moreover, the angle 2θ was also assumed to equal $\pi/2$.

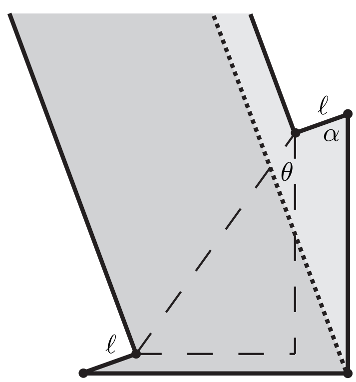
The above result was achieved via elliptic function theory, collecting expressions for the data (g, dh) that depend on the punctured torus. Using the Implicit Function Theorem, I also showed that this surface, $S(1, \pi/2)$, enjoys a toroidal deformation. That is, the following was shown in [6]:

Theorem 2. *Let $\mathbb{C}/\Lambda_\phi$ denote the rhombic torus whose lattice is generated by $\{1, e^{i\phi}\}$. Then, for ϕ sufficiently close to $\pi/2$, there exists $\theta \in (0, \pi/2)$ and points a_i so that $\mathcal{R} = \mathbb{C}/\Lambda_\phi - \{a_1, a_2, a_3, a_4\}$ immerses into $\mathbb{R}^3$ as $S(1, 2\theta)$*

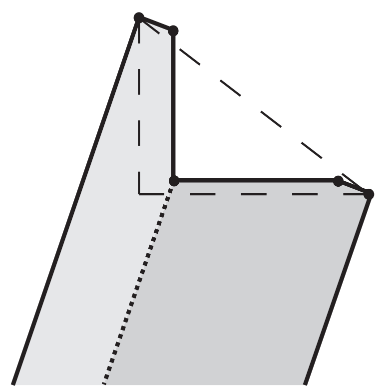
Establishing the existence of these perturbed surfaces involves solving a 2-dimensional period problem. This was accomplished by perturbing the 1-dimensional period problem associated to the much more symmetric surface $S(1, \pi/2)$. It should be noted that these results do not establish the existence of $S(1, 2\theta)$ for θ close to $\pi/4$; this was first obtained by Hauswirth-Traizet [8]. Rather, this theorem only establishes the existence of *possibly* perturbed, genus-1 Scherk surfaces.

Possible Limits of $S(1, 2\theta)$. Using Theorems 1 and 2, we conclude that a family of conformally equivalent gdh and $(1/g)dh$ flat structures exists. Moreover, the proof of Thoerem 2 implies that this family can be parameterized by an analytic curve $\gamma \subset \mathcal{P} \subset \mathbb{R}^3$, where $\mathcal{P}$ is a half-infinite slab. Appealing to Sullivan's Local Euler Characteristic Theorem, (see [7]), it can be shown that this solution set necessarily extends to the boundary of $\mathcal{P}$.

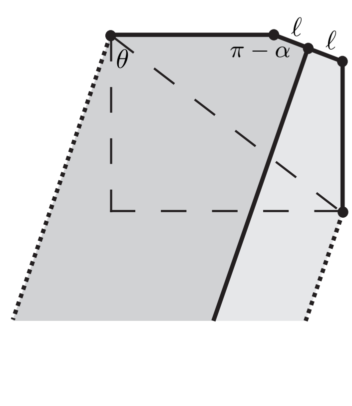
Extremal length arguments involving the gdh and $(1/g)dh$ flat structures restrict where the curve γ can intersect $\partial\mathcal{P}$. Quarters of the flat structures are depicted below



(1/g)dh



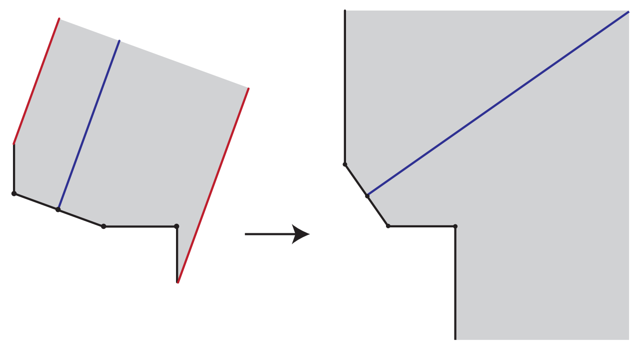
gdh



We depict two versions of the gdh flat structure; the latter is obtained from the former by cutting along the dotted line and then gluing along the solid line.

These domains feature three undetermined parameters, ℓ, θ, and α. After using results of Meeks-Rosenberg [14] and Rosenberg-Toubiana [16] to facilitate various extremal length arguments, it can be shown that the only possible limiting flat structures are given by $(\ell^*, 0, \alpha^*)$ where $\ell^* \in (0, \infty)$ and $\alpha^* \in (0, \pi/2)$. All other possible limits either outright violate the period condition or the conformal equivalence between the full gdh and $(1/g)dh$ flat structures.

The following limiting gdh flat structure is what necessarily results:



The above can be recognized as one quarter of the gdh flat structure for the singly periodic, genus-1 helicoid "on its side." All together, I established the following

Theorem 3. *Given any* $\theta \in (0, \pi/2)$ *there exists* $S(1, 2\theta)$. *Moreover, as* $\theta \to 0$ *the surfaces* $S(1, 2\theta) \to \mathcal{H}(1)$ *in the pointed Gromov-Hausdorff sense.*

Finally, the embeddedness of the surfaces $S(1, 2\theta)$ is achieved by a standard application of the maximum principle (see [5] for example).

4. Future Work

The existence of an embedded, non-periodic, genus-one helicoid was proven in [19]; it was obtained by perturbing the singly periodic genus-one helicoid via screw motions. The surfaces so produced are invariant under a vertical screw motion with angle $2\pi n$ for some $n \in \mathbb{R}$, and so are labeled $\mathcal{H}_n(1)$. The fact that $\lim_{\theta \to 0} S(1, 2\theta) = \mathcal{H}(1)$ holds suggests that it may possible to further perturb or distort the surfaces $S(1, 2\theta)$, producing new surfaces $S(1, 2\theta)_n$ so that $\lim_{\theta \to 0} S(1, 2\theta)_n = \mathcal{H}_n(1)$. Taking a diagonal subsequence would allow one to obtain the genus-one helicoid as a new limit.

It might also be possible to generalize the methods used to obtain $\mathcal{H}(1)$ as a limit of perturbed, genus-one Scherk surfaces to produce singly periodic helicoids of arbitrary genus, which we have notated as $\mathcal{H}(k)$. As it is unknown whether or not these surfaces exist, this would constitute a significant accomplishment. Moreover, once $\mathcal{H}(k)$ is established for arbitrary $k > 1$, one may attempt to generalize the screw-motion techniques of [19] to produce genus-k helicoids.

These generalizations present various interesting and important challenges. In particular, the elliptic function theory underlying the uniqueness proof for $S(1, \pi/2)$ will need to be rephrased in terms of the more general theory of Theta functions. Also, the notion of a "rhombic, genus-k surface" does not exist, and so a suitable analog will need to be developed. Finally, the associated flat structures will necessarily feature more undetermined parameters, increasing the number of possible degenerations. Demonstrating that only one of these degenerations is allowable will then require much more work.

I am also interested in a classical question: When constructing minimal surfaces, is symmetry necessary or merely convenient? Traizet [17] answered this question by constructing minimal surfaces that do not possess any symmetry. However, I am interested in studying this question when the *potential* for symmetry exists. For example, Theorem 1 shows that it is impossible to puncture a square torus *only* with respect to its diagonal symmetries, and not also with respect to its rectilinear symmetries, if one is to immerse it into $\mathbb{R}^3$ as $S(1, 2\theta)$ for some $\theta \in (0, \pi)$. I suspect similar statements are true for the hexagonal torus, as well as for other proposed minimal surfaces whose underlying Riemann surfaces possess some amount of symmetry.

Most generally, my research interests lie in the classification of minimal surfaces. This includes producing limits of known examples through different techniques, such as adding handles, perturbing data, or even gluing together various examples. I would like to continue learning about these different techniques, and I look forward to collaborating with other mathematicians in or around this area.

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