Almost Hermitian and Kähler structures on product manifolds

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It's my great pleasure to give a talk to you about our study.

Falcitelli-Farinola-Salamon (1994) said:

The theory of Kähler manifolds unites some of the most interesting features of complex and Riemannian geometry and a vital branch of differential geometry. However, the need for a wider understanding of the more general class of almost Hermitian manifolds has been emphasized by recent directions of research.

It's my purpose to:

(1) Survey: almost Hermitian, Kähler, almost quaternionic Hermitian and quaternionic Kähler structures constructed on $M \times \mathbb{I}$, where M is a manifold with almost contact metric structures and $\mathbb{I}$ is an open interval.

(2) Application; the notion of a unitary-symmetric Kähler manifold and its geometric properties.

(3) Investigation; almost Hermitian structures on the products of two almost contact metric and Sasakian manifolds.

1. Introduction:

Meaning of research; One can construct, by means of a natural change of the product metric, almost Hemitian structures on product manifolds of two almost contact metric manifolds. Since many almost contact metric and Sasakian structures are now found, this technique enables us to provide various kinds of almost Hermitian structures.

History; In the later half of 1950's, Japanese geometers mainly initiated to study certain types of almost Hermitian geometry, motivated by finding out the nearly Kähler structure in S^6 and the almost Kähler structure on the tangent bundle of a Riemannian manifold.

In 1960, **Kotô** established inclusion relations between various classes. Unfortunately, in this early period, it was a weak point for almost Hermitian geometry to hold two kinds of examples only, mentioned above.

In 1971, **Gray** pointed out that nearly Kähler geometry corresponds to weak holonomy group $U(n)$. Consequently many interesting theorems about the topology and geometry of nearly Kähler manifolds were proved. Then from view point of weak holonomy, **Gray-Hervella** (1980) gave a classification of almost Hermitian manifolds in terms of the covariant derivatives of the fundamental forms. **Falcitelli-Farinola-Salamon** (1994) combined the theory of **Gray-Hervella** and **Tricceri-Vanhecke** into a representation-theoretic framework. In this direction, **Cabrera -Swann** systematically have studied the interaction between these classes when one has an almost hyper-Hermitian structure and in general dimension found at most 167 different almost hyper-Hermitian structures. However, these theories of classification are rigorous but algebraic. Therefore it needs that explicit examples of various type support now the bases of their theories.

2. Preliminaries

The starting point was to write down the Kähler metric

of the complex space form in terms of the usual polar coordinate

Examples 3.1, 3.2).

The tools are almost contact metric and Sasakian structures

(ϕ, ξ, η, g).

2.1 Almost contact metric and Sasakian structures

Let M be an odd-dimensional differentiable manifold.

Definition 2.1. An **almost contact metric structure** on M is by definition a pair (Σ, g) of an almost contact structure $\Sigma = (\phi, \xi, \eta)$ and a Riemannian metric g, where ϕ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form, satisfying the following conditions : $\forall$ vector fields X and Y on M

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \qquad g(X, \xi) = \eta(X). \tag{2.2}$$

Definition 2.2. If (Σ, g) satisfies

$$d\eta(X,Y) = g(\phi X, Y) \tag{2.3}$$

$\forall$vector fields X, Y on M, then (M, Σ, g) is called **a contact Riemannian manifold**. Let M be an almost contact manifold and define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X + f\frac{d}{dt}) = \phi X - f\xi(X) + \eta(X)\frac{d}{dt} \tag{2.4}$$

$\forall$ vector field X of M. An almost contact structure is said to be **normal** if J is integrable.

Definition 2.3. An almost contact metric structure (Σ, g) is called **Sasakian** if furthermore

$$(\nabla_X \phi)(Y) = \eta(Y)X - g(X,Y)\xi \tag{2.5}$$

$\forall$ vector fields X, Y on M.

Definition 2.4. Suppose that a differentiable manifold admits three almost contact structures $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)})$, p=1,2,3 satisfying

$$\eta_{(p)}(\xi_{(q)}) = \delta_{pq}, \tag{2.6}$$

$$\phi_{(p)}\xi_{(q)} = -\phi_{(q)}\xi_{(p)} = \xi_{(r)}, \quad \eta_{(p)} \circ \phi_{(q)} = -\eta_{(q)} \circ \phi_{(p)} = \eta_{(r)},$$

$$\phi_{(p)}\phi_{(q)} - \xi_{(p)} \otimes \eta_{(q)} = -\phi_{(q)}\phi_{(p)} + \xi_{(q)} \otimes \eta_{(p)} = \phi_{(r)}$$

for $\varepsilon(p, q, r) = 1$ where $\varepsilon(p, q, r) = 1$ means that (p, q, r) is a cyclic permutation of $(1, 2, 3)$. Then $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}), p = 1, 2, 3$, is called an **almost contact 3-structure**.

Definition 2.5. A Riemannian metric g is said to be associated to the 3-structure if it satisfies

$$g(\phi_{(p)}X, \phi_{(p)}Y) = g(X, Y) - \eta_{(p)}(X)\eta_{(p)}(Y), p = 1, 2, 3 \tag{2.7}$$

$\forall$ vector fields X, Y on M. In such a manifold with an almost contact 3-structure there always exists a Riemannian metric g satisfying (2.7), and $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1, 2, 3$ is called an **almost contact metric 3-structure**.

Definition 2.6. An almost contact metric 3-structure $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g)$, $p = 1, 2, 3$ is called a **Sasakian 3-structure** if each $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g)$ is a Sasakian structure. Then $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$ are orthonormal vector fields, satisfying

$$[\xi_{(p)}, \xi_{(q)}] = 2\xi_{(r)} \tag{2.8}$$

for $\varepsilon(p, q, r) = 1$. Such a manifold with a Sasakian 3-structure is called a **3-Sasakian manifold**, represented by M^{3S} for simplicity.

3. Almost Hermitian and Kähler structures on $M \times \mathbb{I}$ and unitary-symmetric Kähler manifolds

3.1 Almost Hermitian and Kähler structures on $M \times \mathbb{I}$

Let (M, ϕ, ξ, η, g) be an almost contact manifold of dimension $2p+1$ and $\mathbb{I}$ be $\mathbb{R}$ or an open interval. Then, we introduce an almost complex structure J on the product manifold $M \times \mathbb{I}$, defined in such a way that

$$J(X+T) = \phi X + \frac{1}{\lambda} dt(T)\xi - \lambda\eta(X)\frac{d}{dt} \tag{3.1}$$

$\forall$ vector field X of M, $\forall$vector field T of $\mathbb{I}$, where λ is a positive (or negative) function on $M \times \mathbb{I}$.

Further, we define a Riemannian metric by

$$G(X+T, Y+T') = \alpha g(X,Y) + \beta\eta(X)\eta(Y) + dt^2(T,T') \tag{3.2}$$

$\forall$ vector fields X, Y of M, $\forall$ vector fields T, T' of $\mathbb{I}$, where α, β are functions on $\mathbb{I}$, satisfying

$$\alpha > 0, \qquad \alpha + \beta > 0. \tag{3.3}$$

Then it is easily seen that (J, G) is an almost Hermitian structure on the product $M \times \mathbb{I}$ if and only if their functions satisfy

$$\lambda = \pm\sqrt{\alpha+\beta}. \tag{3.4}$$

Then by summing up these results, we have the following propositions.

Proposition 3.1. *Let* (M, ϕ, ξ, η, g) *be an* **almost contact metric manifold**. *Let* $\alpha = \alpha(t)$, $\beta = \beta(t)$ *are functions on* $\mathbb{I}$, *satisfying (3.3). Then,* (J, G) *is an* **almost Hermitian** *structure on* $M \times \mathbb{I}$. *Moreover,* (J, G) *is is a* **Hermitian structure** *if and only if* (ϕ, ξ, η, g) *is a* **normal almost contact metric** *structure.*

Proposition 3.2. *Let* (M, ϕ, ξ, η, g) *be a* **contact Riemannian** *manifold. Then the almost Hermitian structure constructed such as Proposition 3.1 is* **almost Kählerian** *if and only if* **the functions** α **and** β **satisfy**

$$\beta = \frac{1}{4}(\alpha')^2 - \alpha, \tag{3.5}$$

where we denote by $\alpha' = \frac{d\alpha}{dt}$.

Proposition 3.3 (**Ejiri, Watanabe**). *Let* (M, ϕ, ξ, η, g) *be a* **Sasakian** *manifold. Then the Hermitian structure constructed such as Proposition 3.2 is Kählerian if and only if* **the function** α **is an increasing function, satisfying (3.5).**

If we put $\alpha = f^2$, then $\beta = f^2(f'^2 - 1)$

3.2 Application: Unitary-symmetric Kähler manifolds

Definition 3.1. A Kähler manifold (M, J, g) of complex dimension n is said to be **unitary-symmetric at a point** m **of** M if the linear isotropy group at m of automorphisms (i.e., holomorphic isometries) of M is the unitary group $U(n)$.

Let us fix some notations for explaining some results. Let $m \in M$. We define δ to be the distance from the origin O of the tangent space $T_m(M)$ at m to the fist tangential conjugate locus Q_m. Define $\tilde{B}_\delta = \{X \in T_m(M) | |X| < \delta\}$, where $|X| = \sqrt{g_m(X, X)}$. Then it is clear $\tilde{B}_\delta$ becomes a Riemannian manifold equipped with metric $exp_m^* g$ since the exponential map at m $exp_m : \tilde{B}_\delta \to M$ is non-singular. Then Watanabe (1988) have the following.

Theorem 3.4. *Let (M, J, g) be a complete, connected, simply-conected Kähler manifold of complex dimension $n \geq 2$ and m be a point of M. Then the the following conditions (I) and (II)are equivalent each other:*

(I) (M, J, g) *is unitary-symmetric at m.*

(II) *The metric $exp_m^* g$ and the fundamental form $exp_m^*\Omega$, pulled back under the exponential mapping exp_m, are given by*

$$(3.6)\qquad \begin{aligned} exp_m^* g &= dr^2 + f(r)^2 d\Theta^2 + f(r)^2 (f'(r)^2 - 1)\eta \otimes \eta, \\ exp_m^*\Omega &= 2f(r)f'(r)\eta \wedge dr + f(r)^2\Psi, \end{aligned}$$

on the punctured ball $\tilde{B}_\delta - \{O\}$ of radius δ in $T_m(M)$, *where* f *is a* C^∞ *odd function on* $(-\delta, \delta)$ *such that* $f'(0) = \frac{df}{dt}(0) = 1$. *Here we assume that* δ *is infinite when* M *is non-compact, and we denote by* (r, Θ) *the usual polar coordinate system of* $\mathbb{C}^n \equiv T_m(M)$, *by* $(d\Theta^2, \phi, \xi.\eta)$ *the standard Sasakian structure on the unit sphere* S^{2n-1} *in* $T_m(M)$, *and set* $\Psi(X, Y) = d\Theta^2(\phi X, Y)$.

Example 3.1. Let (M, J, g) be a complex space form, endowed with the canonical Kähler metric $d\sigma^2$ of constant holomorphic curvature $4k$. Then, from **Tachibana's** result,the Kähler metric is given, by using the function $f(r)$, in (II) of Theorem 3.4, as follows:

(1) for $k = 0, \quad M = \mathbb{C}^n, \quad f(r) = r,$

(2) for $k > 0, \quad M = \mathbb{C}P^n, \quad f(r) = \frac{1}{\sqrt{k}} sin\sqrt{k}r.$

(3) for $k < 0, \quad M = \mathbb{C}H^n, \quad f(r) = \frac{1}{\sqrt{-k}} sinh\sqrt{-k}r.$

In fact, the Kähler metric $d\sigma^2$ for the case (2) is give by

$$d\sigma^2 = sin^2 r d\Theta^2 + sin^2 r(cos^2 r - 1)\eta \otimes \eta + dr^2.$$

Example 3.2. Let us consider a Kähler metric $g = (g_{\alpha\bar{\beta}})$ in $\mathbb{C}^n$, given by the potential function

$$h(t) = \int_0^t \frac{1}{s} log(1+s) ds$$

for $t = \Sigma z^\alpha \bar{z}^\alpha$. Then it is complete and has positive curvature (**Klembeck**). By **Itoh's** result, we have

$$\begin{aligned} exp_0^* \, g &= dr^2 + 2log(cosh\ r)d\Theta^2 + (tanh^2 - 2log(cosh\ r))\eta \otimes \eta, \\ exp_0^* \, \Omega &= 2(tanh\ r)\eta \wedge dr + 2log(cosh\ r)\Psi, \end{aligned}$$

where exp_0^* is the differential of exponential mapping at the origin 0 of $\mathbb{C}^n$.

Theorem 3.5. *Let* (ds^2_{can}, J_{can}) *be the canonical Kähler structure on the complex projective n-space* $\mathbb{C}P^n$. *Then, for a sufficient small positive number* ε *there exists a* **one-parameter family of Kähler structures** $(ds^2_a, J_{can})(\varepsilon < a < \varepsilon)$ *on* $\mathbb{C}P^n$, satisfying

(1) *For* $a = 0$, $ds^2_a = ds^2_{can}$.

(2) *For diffeent values* $a, b \in (-\varepsilon, \varepsilon)$ *and for each* $\lambda > 0$, $ds^2_a \neq \lambda ds^2_b$.

(3)*For any* a $(\varepsilon < a < \varepsilon)$, *there is a point* m *of* $\mathbb{C}P^n$ *and a constant* $\ell(a)$ *such that* $(\mathbb{C}P^n, ds^2_a)$ *is a* SC^m **manifold**, *that is, all geodesics issuing from* m *are simple closed ones with length* $\ell(a)$.

3.3. Quaternionic Kähler structures on $M^{3S} \times \mathbb{I}$

Definition 3.2. An almost hypercomplex structure on a manifold M of dimension $4m$ is by definition a triple $H = (J_{(p)}), p = 1, 2, 3$ of almost complex structure, satisfying

$$J_{(p)} J_{(q)} = J_{(r)} \tag{3.7}$$

for $\varepsilon(p, q, r) = 1$(see Ishihara (1974), Alekseevsky-Marchiafava(1993) for detail).

Definition 3.3 A Riemannian manifold is called a **quaternionic Kähler manifold** if the holonomy group is a subgroup of $Sp(n) \cdot Sp(1)$. Then it is known that in a quaternionic Kähler manifold there exist an almost hypercomplex structure $J_{(p)}, p = 1, 2, 3$ such that in any local coodinate neighborhood U satisfies

(3.8)
$$\begin{aligned} \nabla_X J_{(1)} &= \gamma(X) J_{(2)} - \beta(X) J_{(3)}, \\ \nabla_X J_{(2)} &= -\gamma(X) J_{(1)} + \alpha(X) J_{(3)}, \\ \nabla_X J_{(3)} &= \beta(X) J_{(1)} - \alpha(X) J_{(2)} \end{aligned}$$

for any vector field X on U, where ∇ is the Levi-Civita connection of the Riemannian metric , and α, β, γ are certain local 1-forms defined in U and vice versa. In particular, if all α, β, γ for each U are vanishing, then the structure is called **hyper-Kähler.**

Let $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1, 2, 3$, be an almost contact metric 3-structure on a manifold M of dimension $4m+3$. By $\mathbb{I}$ we denote $\mathbb{R}$ or some open interval in $\mathbb{R}$. For a positive function λ on $\mathbb{I}$, we define an almost hypercomplex structure $\tilde{J}_{(p)}, p = 1, 2, 3$, on $M \times \mathbb{I}$ by (3.7), that is,

$$\tilde{J}_{(p)} = \begin{pmatrix} \phi_{(p)} & -\lambda\eta_{(p)} \\ \\ \lambda^{-1}\xi_{(p)} & 0 \end{pmatrix}, p = 1, 2, 3. \tag{3.8}$$

Let α, β be real valued function on $\mathbb{I}$, satisfying

$$\alpha(t) > 0, \qquad \alpha(t) + \beta(t) > 0. \tag{3.9}$$

Then, we give a Riemannian metric on $M \times \mathbb{I}$ by

$$\tilde{g} = \alpha(t) g + \beta(t) \sum_{p=1}^{3} \eta_{(p)} \otimes \eta_{(p)} + dt^2 \tag{3.10}$$

where dt^2 is the usual metric on $\mathbb{I}$. Then by (3.8) and (3.10) **Nakashima-Watanabe** showed the following.

Proposition 3.6. *Let $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1,2,3$ be an almost contact metric 3-structure on a manifold M of dimension $4m+3$ and λ a function on $\mathbb{I}$. Let α, β be real valued function on $\mathbb{I}$, satisfying (3.9). $(\tilde{J}_{(p)}, \tilde{g}), p = 1,2,3$ is an almost quaternionic Hermitian structure on $M \times \mathbb{I}$ if and only if $\lambda = \pm\sqrt{\alpha+\beta}$.*

Next, let $(M, \phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1,2,3$ be a 3-Sasakian manifold of dimension $4m+3$ and α, β be real valued function on $\mathbb{I}$, satisfying (3.15) and $\lambda = \sqrt{\alpha+\beta}$. A directly compute of $\tilde{\nabla}\tilde{J}_{(p)}$, shows that the functions

α and β

$$2\sqrt{\alpha+\beta} = \frac{d\alpha}{dt}, \qquad \alpha\frac{d\beta}{dt} = 4\beta\sqrt{\alpha+\beta}, \tag{3.11}$$

then it is easily seen that $(\tilde{J}_{(p)}, \tilde{g}), p = 1, 2, 3$ is a quaternionic Kähler structure. Moreover, putting $\alpha = f^2$ and hence $\beta = f^2(f' - 1)$, we see that the metric (3.16) reduces to

$$\tilde{g} = f(t)^2 g + f^2(f'^2 - 1)\sum_{p=1}^{3} \eta_{(p)} \otimes \eta_{(p)} + dt^2 \tag{3.12}$$

Proposition 3.7. *Let* $(M, \phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1, 2, 3$ *be a 3-Sasakian manifold. An almost quaternionic Hermitian structure constructed on* $M \times \mathbb{I}$ *such as Proposition 3.6 is quaternionic Kähler if and only if satisfies the ODE*

$$ff'' - (f')^2 + 1 = 0. \tag{3.13}$$

with the conditions $f > 0$ *and* $\frac{df}{dt} > 0$ *on* $\mathbb{I}$.

Now, putting $p = \frac{df}{dt} = f'$, we have $f'' = p\frac{dp}{df}$, from which (3.13) reduces to

$$p^2 - 1 = kf^2,$$

where k is constant. Recall that $p = f'(t)$ and that $f(t) > 0$ and $f'(t) > 0$. Thus we have, up to a motion of parameter t, a generalization of the result in **Boyer-Galicki-Mann**, since $f' = 1$ in the case $k = 0$.

Theorem 3.8. *Let* $(M, \phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1, 2, 3$ *be a* **3-Sasakian manifold**. *Let* f *be a real valued function, satisfying the ODE (3.13). Then we have*

(1) *If* $f(r) = r$, *then the product* $M \times \mathbb{R}^+$ *with the cone metric in (3.12) is hyper-Kähler.*

(2) *If* $f(r) = \frac{1}{\sqrt{-k}} sinh(\sqrt{-k}t)$, *then the product* $M \times \mathbb{R}^+$ *with the cone metric in (3.12) is quaternionic Kähler, where* k *is a negative constant.*

(3) *If* $f(t) = \frac{1}{\sqrt{k}} sin(\sqrt{k}t)$, *the the product* $M \times (0, \frac{\pi}{\sqrt{k}})$ *with the metric in (3,12) is quaternionic Kähler, where* k *is a positive constant.*

4. Almost Hermitian structures on various products

In this place, let us consider the product manifold $M \times M'$ of two almost contact metric manifolds (M, Σ, g) and (M', Σ', g') respectively.

First, we introduce **a class of almost complex structure** $J_{(\rho,\sigma)}$ on the product manifold $M \times M'$:

$$(4.1)\qquad \begin{aligned} J(X+X') = {} & \phi X - \{\frac{\rho}{\sigma}\eta(X) + \frac{\rho^2+\sigma^2}{\sigma}\eta'(X')\}\xi \\ & +\phi' X' + \{\frac{1}{\sigma}\eta(X) + \frac{\rho}{\sigma}\eta'(X')\}\xi' \end{aligned}$$

for any vector fields X, Y of M and any vectors fields X', Y' of M', where ρ, σ are functions on $M \times M'$ and further σ is a positive (or negative) function.

Next, we define a Riemannian metric $G_{(\alpha,\beta,\gamma,\delta,\mu)}$ on $M \times M'$, by

$$(4.2)\quad G_{(\alpha,\beta,\gamma,\delta,\mu)} = \alpha g + \beta\eta\otimes\eta + \mu(\eta\otimes\eta' + \eta'\otimes\eta) + \gamma g' + \delta\eta'\otimes\eta'$$

for any vector fields X, Y of M and any vector fields X', Y' of M', where α, β, γ, δ and μ are functions on $M \times M'$, satisfying

$$(4.3)\quad \alpha > 0,\quad \alpha+\beta > 0,\quad \gamma > 0,\quad \gamma+\delta > 0,\quad (\alpha+\beta)(\gamma+\delta) > \mu^2.$$

Then it is easily seen from (2,1) and (2,2) that $(J_{(\rho,\sigma)}, G_{(\alpha,\beta,\gamma,\delta,\mu)})$ is an almost Hermitian structure on the product $M \times M'$ if we define ρ and σ by

$$(4.4)\qquad \rho = \frac{\mu}{\alpha+\beta},\qquad \sigma = \pm\frac{\sqrt{(\alpha+\beta)(\gamma+\delta)-\mu^2}}{\alpha+\beta}.$$

In this section we denote the almost Hermitian structure $(J_{(\rho,\sigma)}, G_{(\alpha,\beta,\gamma,\delta,\mu)})$ simply by (J, G).

By a direct computation using (2.1) and (2,2) again, one can check that **the fundamental 2-form** Ω **of** (J, G) is very simply as follows:

$$\Omega = \alpha\Psi + \gamma\Psi' + \sqrt{(\alpha+\beta)(\gamma+\delta)-\mu^2}(\eta \wedge \eta'), \tag{4.5}$$

where we denote by $\Psi(X, Y) = g(\phi X, Y)$ and $\Psi'(X', Y') = g'(\phi' X', Y')$ for any vector fields X, Y of M and any vectors fields X', Y' of M', although J and G are so complicatedly defined.

Here, suppose that all $\alpha, \beta, \gamma, \delta$ and μ in (4.2) are constants, satisfying (4.3). So, if we define ρ and σ by (4.4), then the almost complex structure J is integrable. Thus, we have a generalization of **Caprusi's** result.

THEOREM 4.1. *Let us consider two* **normal almost contact metric manifolds** (M, Σ, g), (M, Σ, g). *Suppose that all* $\alpha, \beta, \gamma, \delta$ *and* μ *in (4.2) are constants, satisfying (4.3). Then the Hermitian structure* (J, G) *on the product* $M \times M'$ *is a* **Kähler structure** *if and only if both factors are* **cosymplectic**.

In particular, when $\alpha = 1, \beta = 0, \gamma = 1, \delta = 0$ and $\mu = 0$, the almost Hermitian metric (J, G) is a Hermitian structure which J is given by **Morimoto** (1963) and G is product.

4.1. Open question of Blair and Oubiña

It is natural to find out some conditions for almost Hermitian structures defied on the product of two Sasakian manifolds to be Kählerian. This is the open question due to **Blair-Oubiña**. But such a case, in general, is incorrect as is seen from **Calabi-Eckmann** manifold, which is not able to admit any Kähler metric.

Our motivation, that introduces (J, G) given in (4.1) and (4.2), is to solve the open question of **Blair and Oubiña** and to construct almost Hermitian structures of various type as many as possible.

4.2 Almost Hermitian structures on the product $M^{3S} \times M'$

Let $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1, 2, 3$ be an almost contact metric 3-structure on a manifold M of dimension $4m + 3$ and (ϕ', ξ', η', g') an almost contact metric structure on a manifold M'. Then we define three kind of almost complex structures

$J_{(p)}, p = 1, 2, 3$ on the product manifold $M \times M'$: $\forall$ functios ρ, σ

(4.6)

$$\begin{aligned} J_{(p)}(X + X') = \; & \phi_{(p)}X - \{\frac{\rho}{\sigma}\eta_{(p)}(X) + \frac{\rho^2 + \sigma^2}{\sigma}\eta'(X')\}\xi_{(p)} \\ & + \phi' X' + \{\frac{1}{\sigma}\eta_{(p)}(X) + \frac{\rho}{\sigma}\eta'(X')\}\xi', \quad p = 1, 2, 3. \end{aligned}$$

for any vector field X of M^{3S} and any vector field X' of M', where

σ is a positive (or negative) function on $M \times M'$. Next, we define a Riemannian metric on $M \times M'$, by

(4.7)

$$G_{(p)} = \alpha g + \beta \sum_{p=1}^{3} (\eta_{(p)} \otimes \eta_{(p)}) + \mu \sum_{p=1}^{3} (\eta_{(p)} \otimes \eta' + \eta' \otimes \eta_{(p)}) + \gamma g' + \delta \eta' \otimes \eta'.$$

$p = 1, 2, 3$ for any vector fields X, Y of M^{3S} and any vector fields X', Y' of M', where α, β, γ, δ and μ are functions on $M \times M'$, satisfying the conditions (4.3). Then one can see that for each p=1,2,3 $(J_{(p)}, G_{(p)})$ given by (4.4) is an almost Hermitian structure on the product $M^{3S} \times M'$. Then we ask the following two problems.

Problem 4.1. How almost Hermitian geometry can one extend on $M^{3S} \times M'$ with three kinds of almost Hermitian structures $(J_{(p)}, G_{(p)})$, $p = 1, 2, 3$ in contrast with quaternionic geometry ?

Problem 4.2. Let $(\phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1, 2, 3$ be a Sasakian 3-structure on a manifold M^{3S} of dimension $4m + 3$ and (ϕ', ξ', η', g') a Sasakian structure on a manifold M' of dimension $2q + 1$. Can one construct quaternionic Kähler structures on $M^{3S} \times M'$ when q is an even number ?

4.3 Almost Hermitian structures on the product $M^{3S} \times M'^{3S}$

Further, by extending Problem 4.2, we ask the following.

Problem 4.3. Let $(M, \phi_{(p)}, \xi_{(p)}, \eta_{(p)}, g), p = 1, 2, 3$ be a 3-Sasakian manifold M and $(M, \phi'_{(q)}, \xi'_{(q)}, \eta'_{(q)}, g'), q = 1, 2, 3$ another 3-Sasakian manifold. How almost Hermitian geometry can one extend on $M \times M'$ with nine kinds of almost complex structures ?

On the other hand, it is known that there exists a homogeneous nearly Kähler structure on $S^3 \times S^3$. This structure on $S^3 \times S^3$ is related to a general construction due to **Ledger and Obata**: If G is any compact simple Lie group, then $G \times G$ is a Riemannian 3-symmetric. As S^3 admits a Sasakian 3-structure, we ask for the following.

Problem 4.4. Can one realize the homogeneous nearly Kähler structure on $S^3 \times S^3$ in terms of Sasakian 3-structures on S^3 ?

Problem 4.5. Can one construct the other non-homogeneous nearly Kähler structure on $S^3 \times S^3$?

In particular, from Riemannian geometric point of view , we are very interested in the following.

Problem 4.6. (The **Hopf conjecture**): Does $S^2 \times S^2$ admit a metric with positive sectional curvature ? Can one, by making use of a similar change of metric to (4.7), construct a metric with positive sectional curvature on $S^3 \times S^3$?

Thank you

Thank you for your kind attention !!