Finsler metrics (Flag Curvature)

Miguel Angel Javaloyes and Miguel Sánchez

Universidad de Granada

Seminario del departamento de Geometría y topología
16 de diciembre de 2009

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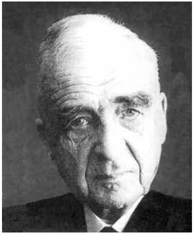

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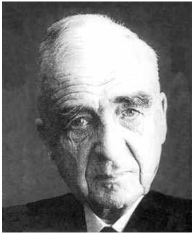

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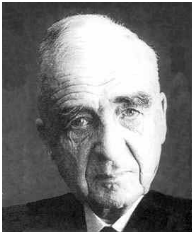

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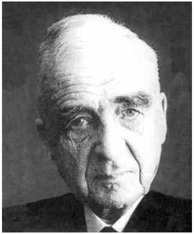

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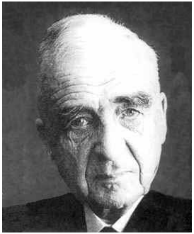

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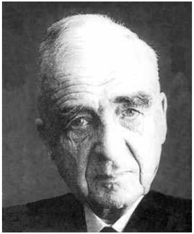

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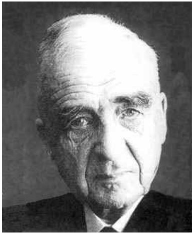
- F^2 is C^1 on TM.



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Closed Geodesics

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- S^2 with a Riemannian metric admit infinite many closed geodesics (Franks (92) and Bangert (93))

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S.S. Chern (1911-2004)

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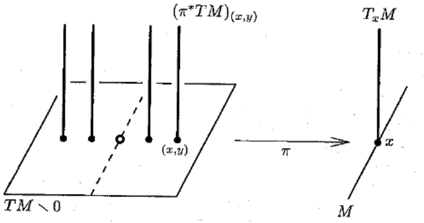

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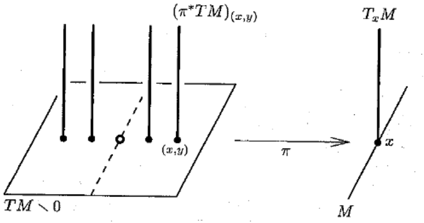

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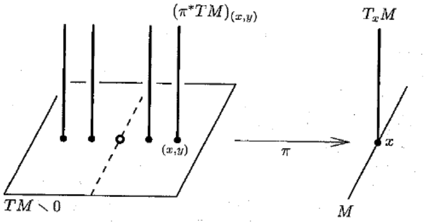

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$$N_j^i(x,y) := \gamma^i_{jk} y^k - \frac{1}{F} A^i_{jk} \gamma^k_{rs} y^r y^s$$

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$$\gamma^i{}_{jk}(x,y) = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j}\right), A_{ijk}(x,y) = \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4}\frac{\partial^3(F^2)}{\partial y^i \partial y^j \partial y^k},$$

Covariant derivatives

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- The Chern connection gives two different covariant derivatives:

$$D_T W = \left(\frac{\mathrm{d}W^i}{\mathrm{d}t} + W^j T^k \Gamma^i_{jk}(\gamma, T)\right)\frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} \quad \textit{with ref. vector } T,$$

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E. CARTAN (1861-1940)

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Hanno Rund 1925-1993, South Africa

Curvature 2-forms of the Chern connection

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First Bianchi Identity for R

Luigi Bianchi (1856-1928)

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Luigi Bianchi (1856-1928)

Bianchi Identities

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Second Bianchi identities: very complicated, mix terms in $R_j{}^i{}_{kl}$ and $P_j{}^i{}_{kl}$

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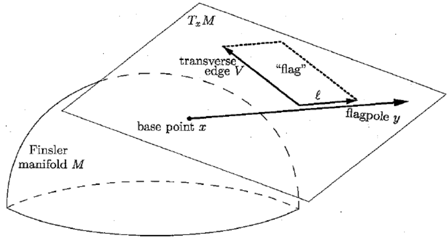
Flag Curvature

We must fix a flagpole y and then a transverse edge V



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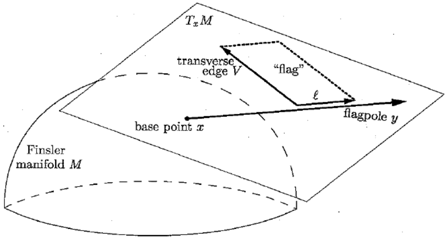
$$K(y,V) := \frac{V^i (y^j R_{jikl} y^l) V^k}{g(y,y)g(V,V) - g(y,V)^2}$$



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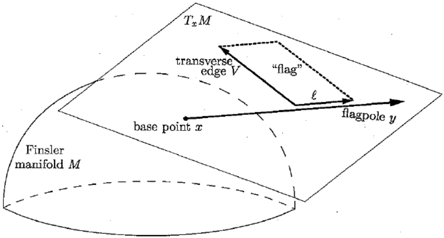

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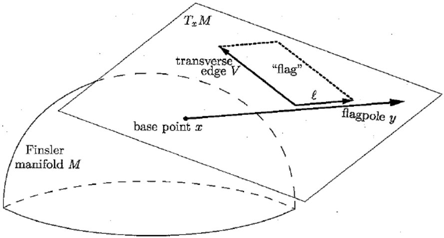
- We can change V by $W = \alpha V + \beta y$, that is, $K(y,W) = K(y,V)$.
- We obtain the same quantity with the other connections (Cartan, Berwald, Hasiguchi...



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- the flag curvature does not depend on the transverse edge!! it is scalar

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- Still no classification (solutions $\sqrt{h} + h(W, v)$ must have a h-Riemannian curvature related with the module of a h-Killing field W)
- Finally they perceive that when considering Zermelo expression of Randers metrics the geometry comes out

Flag constant curvature and stationary spacetimes

- Zermelo metric:

$$\sqrt{\frac{1}{\alpha}g(v,v)+\frac{1}{\alpha^2}g(W,v)^2}-\frac{1}{\alpha}g(W,v),$$

where $\alpha = 1 - g(W,W)$.

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- what about scalar flag curvature?

Schur's Lemma

Theorem

Let M be a Riemannian manifold with dimension ≥ 3. If for every point $x \in M$ the sectional curvature does not depend on the plain, then M has constant sectional curvature.

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- Generalized to Finsler manifolds by Lilia del Riego in her Phd. Thesis in 1973.

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Gauss-Bonnet Theorem

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Suppose M is a 2-dim compact Riemannian manifold with boundary ∂M. Then $\int_M K \, dA + \int_{\partial M} k_g \, ds = 2\pi\chi(M)$,

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S. S. Chern (1911-2004)

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- Bao-Chern (Ann. Math. 1996) extend it to a wider class of Finsler manifolds

David Bao and S. S. Chern

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D. Bao, S.S. Chern and Z. Shen

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- Causality reveals that completeness can be substituted by the condition

$$B^{+}(x,r) \cap B^{-}(x,r) \text{ compact for all } x \in M \text{ and } r > 0$$

(see Caponio-M.A.J.-Sánchez, preprint 09)

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- Again the completeness condition can be weakened.

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Cartan-Hadamard Theorem

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If M is a geodesically complete connected Riemannian manifold of non positive sectional curvature. Then

- *Geodesics do not have conjugate points*
- $\exp_p : T_pM \longrightarrow M$ *is globally defined and a local diffeorphism*
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- Dazord observes that Klingeberg proof works for reversible Finsler metrics in 1968

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- In 2007 S. Brendle and R. Schoen (J. Amer. Math. Soc 2009) prove by using Ricci-flow that there exists a diffeomorphism

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Theorem

A simply connected connected manifold with $\frac{1}{4} < K \leq 1$ is homeomorphic to the sphere.

- In 2004 H. B. Rademacher (Math. Ann.) extends Klingenberg proof to non-reversible Finsler metrics using the hypothesis $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$, where $\lambda = \max\{F(-X) : F(X) = 1\}$
- In 2007 S. Brendle and R. Schoen (J. Amer. Math. Soc 2009) prove by using Ricci-flow that there exists a diffeomorphism
- To obtain Rademacher's result it is enough symmetrized compact balls and bounded reversivility index

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- Toponogov theorem? Problems with angles

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Bibliography

D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, vol. 200 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

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