Complete Flat Surfaces with two Isolated Singularities in $\mathbb{H}^3$.

Francisco Milán López

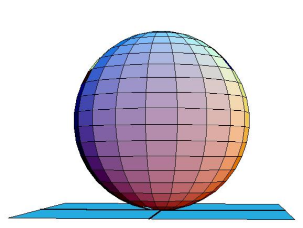
Joint work with Armando V. Corro and Antonio Martínez

Classical Results

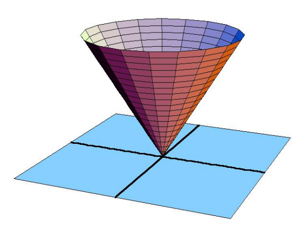
Theorem (Volkov-Vladimirova (1971), Sasaki (1973))

The only complete examples of flat surfaces in $\mathbb{H}^3$ (without singularities) are horospheres and hyperbolic cilinders.

Horosphere



Hyperbolic Cylinder

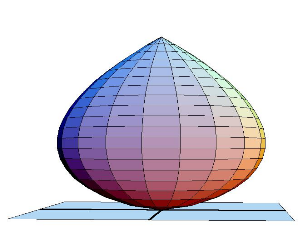


Classical Results

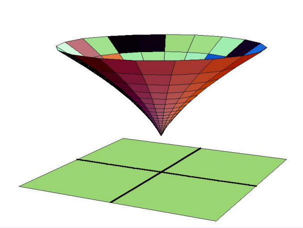
Up to now

The only known examples of complete flat surfaces in $\mathbb{H}^3$ with isolated singularities are rotational ones.

Rotational flat surface



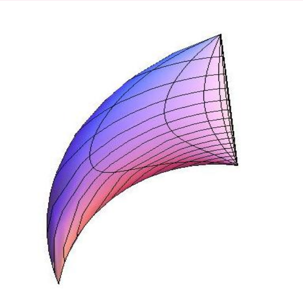
Rotational flat surface



Problem

Existence and characterization of new examples of complete flat surfaces in $\mathbb{H}^3$ with isolated singularities.

Solution with two isolated singularities and one end



- Singularities in a vertical line.
- Coordinates strongly related.

Main Schedule

1. Recent Tools from Weierstrass representation
2. Flat surfaces with n isolated singularities
3. Rotational examples (n = 1)
4. Solved Problem (n = 2)
5. Unsolved Problem ($n > 2$)
6. Canonical examples (n = 2)
7. Characterizations

Recent Tools

- From Bryant's representation for surfaces of mean curvature one in $\mathbb{H}^3$ and our results about improper affine spheres, (Ferrer, Martínez, M, 1996):

Flat surfaces in $\mathbb{H}^3$ admit a Weierstrass representation in terms of meromorphic data, (Gálvez, Martínez, M, 2000).

- Study of ends and singularities.
- Generic behaviour and existence of complete examples with curves of singularities, (Kokubu, Rossman, Saji, Umehara, Yamada, Roitman).
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Weierstrass representation

We consider the upper half-space model of $\mathbb{H}^3$, that is,

$$\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$$

endowed with the metric

$$\langle , \rangle := \frac{1}{x_3^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

of constant curvature -1 and with ideal boundary

$$\mathbb{C}_\infty = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\} \cup \{\infty\} \equiv \mathbb{S}^2.$$

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Let Σ be a $2-$manifold and

$$\psi : \Sigma \longrightarrow \mathbb{H}^3$$

be a flat immersion. Then, the second fundamental form $d\sigma^2$ is definite. In fact, if the induced metric is given by

$$ds^2 = dx^2 + dy^2$$

then

$$d\sigma^2 = \phi_{xx} dx^2 + \phi_{yy} dy^2 + 2\phi_{xy} dxdy$$

with

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1.$$

From this Monge-Ampère equation we obtain holomorphic data for the improper affine spheres and for the flat surfaces.

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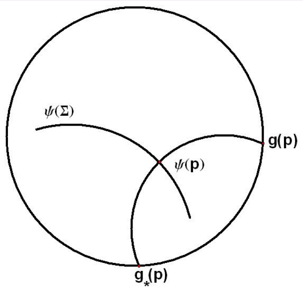
If we consider the Riemann surface $(\Sigma, d\sigma^2)$, then the hyperbolic Gauss maps $g, g_* : \Sigma \longrightarrow \mathbb{C}_\infty \equiv \mathbb{S}^2$ are holomorphic.

These maps associate each point $p \in \Sigma$, with the limit points of the geodesic at $\psi(p)$ orthogonal to $\psi(\Sigma)$.



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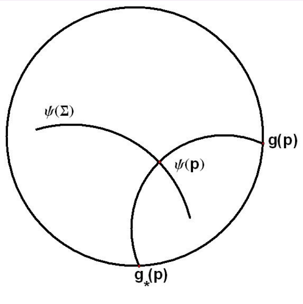

Weierstrass representation

- With these holomorphic data we obtained a conformal representation and different results for flat surfaces in $\mathbb{H}^3$:
  - Description of classical examples.
  - A complete end is conformal to a disk minus a point z_0.
  - g extends to z_0 iff the associated ODE has a regular singularity.
  - A regular end is embedded iff z_0 is not a branch point of g.
- Kokubu, Umehara and Yamada extended our representation to flat fronts, that is, flat surfaces with admissible singularities, $((g, g_*) : \Sigma \longrightarrow \mathbb{S}^2 \times \mathbb{S}^2$ is an immersion), (2004).
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Let g and g_* be non-constant meromorphic functions on a Riemann surface Σ, such that $g(p) \neq g_*(p)$ for all $p \in \Sigma$,

1. all the poles of the 1-form $\frac{dg}{g-g_*}$ are of order 1,
2. $\mathrm{Re} \int_\gamma \frac{dg}{g-g_*} = 0$, for each loop γ on Σ and
3. g and g_* have no common branch points.

If $\xi := c \exp \int \frac{dg}{g-g_*}$, with $c \in \mathbb{C} \setminus \{0\}$.

Then, the map $\psi = (\psi_1, \psi_2, \psi_3) : \Sigma \longrightarrow \mathbb{H}^3$ given by

$$\psi_1 + \mathrm{i}\,\psi_2 = g - \frac{|\xi|^4 (g - g_*)}{|\xi|^4 + |g - g_*|^2}, \qquad \psi_3 = \frac{|\xi|^2 |g - g_*|^2}{|\xi|^4 + |g - g_*|^2}$$

is a singly-valued flat front. Conversely any non-totally umbilical flat front can be constructed in this way.

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From this theorem, one has:

- A harmonic function $u : \Sigma \backslash \mathcal{P}_g \longrightarrow \mathbb{R}$, given by

$$u := \mathrm{Re} \int \frac{dg}{g - g_*},$$

where $\mathcal{P}_g$ is the set of poles of g. (In our examples $\mathcal{P}_g = \emptyset$).

- A holomorphic function $F := \frac{1}{g - g_*}$ on Σ, determined by

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Thus, we can change the data (g, g_*) by (g, u).

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Theorem (Corro, Martínez, M (2010))

Let g be a non-constant meromorphic function on a Riemann surface Σ and u be a harmonic function as above. If

$$ds^2 = \exp(-4u)\left|\exp(4u)(dF + F^2 dg) - \overline{dg}\right|^2$$

is a Riemannian metric, then the map $\psi = (\psi_1, \psi_2, \psi_3) : \Sigma \longrightarrow \mathbb{H}^3$,

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is a well-defined flat immersion, with second fundamental form

$$d\sigma^2 = \exp(-4u)\,|dg|^2 - \exp(4u)\left|dF + F^2 dg\right|^2.$$

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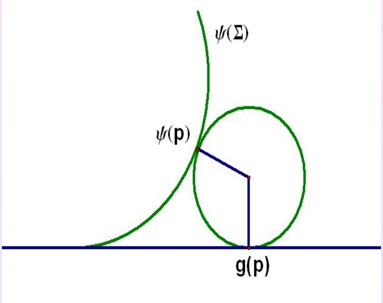
Remark (Corro (2006))

The hyperbolic Gauss map g defines a horospheres congruence and $\exp(2u)/2$ is the radius function of each tangent horosphere to $\psi(\Sigma)$.



Flat surfaces with isolated singularities

Definition

Let Σ be a differentiable surface without boundary, $\psi : \Sigma \to \mathbb{H}^3$ a continuous map and $\mathcal{F} = \{p_1, \cdots, p_n\} \subset \Sigma$ a finite set. We say that ψ is a complete flat immersion with isolated singularities $\psi(p_1), \cdots, \psi(p_n)$, if ψ is a flat immersion in $\Sigma \setminus \mathcal{F}$, but ψ is not C^1 at the points $p_1, \cdots, p_n$, and every divergent curve in Σ has infinite length for the induced (singular) metric.

- Around an (embedded) isolated singularity we have the conformal structure of an annulus (and a convex graph), (Gálvez and Mira (2005)).
- $ds^2 \leq 2\exp(-4u)|dg|^2$ and from the classical result of Huber:

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Let $\psi : \Sigma \to \mathbb{H}^3$ be a complete flat immersion with isolated singularities $\psi(p_1), \cdots, \psi(p_n)$.
Then there is a compact Riemannian surface $\overline{\Sigma}$, n disjoint discs $\mathcal{D}_1, \cdots, \mathcal{D}_n \subset \overline{\Sigma}$ and points $q_1, \cdots, q_m \in \overline{\Sigma} \setminus \{\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n\}$ such that

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endowed with the conformal structure induced by the second fundamental form has the conformal type of

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The points $q_1, \cdots, q_m$ are called the ends of ψ.

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Proposition

Each embedded complete end of a flat surface in $\mathbb{H}^3$ is biholomorphic to a punctured disc and the hyperbolic Gauss map g extends meromorphically to the punctured, that is, the end must be regular (and a convex graph).

Notes of the Proof

(Gálvez, Martínez, M (2000))

$$\kappa = 0 \sim ODE \sim H = 1$$

(Yu, Surfaces of constant mean curvature one in $\mathbb{H}^3$ with irregular ends (2001))

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Theorem

If $\psi : \Sigma \to \mathbb{H}^3$ is a complete flat embedding with a finite number of isolated singularities, then ψ is globally convex.

Corollary

Every complete flat embedding $\psi : \Sigma \to \mathbb{H}^3$ with a finite number of isolated singularities and only one end is a graph over a finitely punctured horosphere and

$$\Sigma \setminus \{p_1, \cdots, p_n\} \cong \mathbb{S}^2 \setminus \{\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n \cup \{q\}\}.$$

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Flat surfaces with isolated singularities

Consequence

The existence of complete flat embedding in $\mathbb{H}^3$ with n isolated singularities and only one end is equivalent to the existence of the appropriate data (g, g_*) or (g, u) on $\mathbb{C} \setminus \{\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n\}$.

Flat surfaces with isolated singularities

First approach

As $ds^2 = 0$ on $\partial\mathcal{D}_j$, $j = 1, \ldots, n$, one has

$$\left|\frac{\exp(4u)(dF + F^2 dg)}{dg}\right| = 1.$$

Then, we tried to recover (g, u) from meromorphic functions $\widetilde{f}, \widetilde{g}$ on $\mathbb{C} \setminus \{\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n\}$ such that $\left|\frac{d\widetilde{f}}{d\widetilde{g}}\right| = 1$ on the boundary. But the way is complicated.

However, this idea gives directly the data for the regular solution of the Monge-Ampère equation $\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1$, in the plane minus n points, (Gálvez, Martínez, Mira (2005)).

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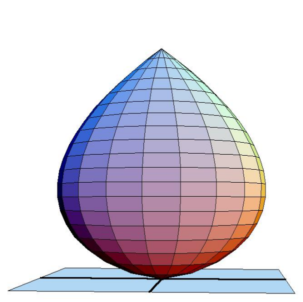
Rotational examples (n = 1)



The half hourglass is a flat complete embedding $\psi : \Sigma \longrightarrow \mathbb{H}^3$, with one isolated singularity and one end.

It has the elementary data

$$g(z) = z, \quad g_*(z) = \frac{a+1}{a-1} z,$$

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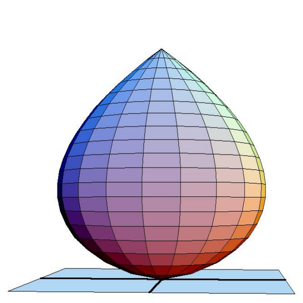

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with

$$z \in \Sigma = \mathbb{D}_r^* = \{z \in \mathbb{C} \;/\; 0 < |z| < r\} \cong \mathbb{C} \setminus \mathcal{D}_1,$$

$4r^{2a} = 1 - a^2$ and $a \in]0, 1[$.

- The singularity is $\psi(\mathbb{S}_r)$, with $\mathbb{S}_r = \{z \in \mathbb{C} \;/\; |z| = r\}$.
- The end is $\psi(0)$.
- The function

$$R(z) = F(z)g(z) = \frac{g(z)}{g(z) - g_*(z)}$$

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Solved Problem (n = 2)

We find the data (g, u) from an apropppriate holomorphic function $R : \mathbb{A}_r^* \longrightarrow \mathbb{C}$, where

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and

$$|g(z)|^2 = \frac{c^2 R(z)}{1 - R(z)}, \quad z \in \mathbb{S}_r,$$

where R is the holomorphic function $F(g - 0)$.

Solution

These conditions determine R, also $g \sim \sqrt{R/1 - R}$ and u from $F = R/g$. Thus, we are going to construct new examples that we will call canonical examples.

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It is more difficult, because the singularities are not in the same vertical line $\{0\} \times \mathbb{R}^+$ and one has different undetermined holomorphic functions

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with $c_j \in \mathbb{C}$, $j = 1, \ldots, n$. (Each R_j is real only on $\partial \mathcal{D}_j$).

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Find a good combination of the above functions.

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Canonical examples $(n = 2)$

In this case, we can find explicitly an appropriate holomorphic function R and this is fundamental in the proofs.

We consider the annular Jacobi theta function in $\mathbb{A}_r$ given by

$$\vartheta_1(z) = \left(1 - \frac{1}{z}\right) \prod_{k=1}^{\infty} (1 - r^{2k} z)(1 - r^{2k}/z).$$

It satisfies

$$\vartheta_1(z) = \overline{\vartheta_1(\overline{z})} = -r^2 z \vartheta_1(r^2 z) = -\frac{1}{z}\vartheta_1(1/z), \ \ \vartheta_1(z/r^2) = -z\vartheta_1(z)$$

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Thus, for any $z_j \in]-1, -r[$, the classical holomorphic bijection $q_j : \mathbb{A}_r \setminus \{z_j\} \longrightarrow \mathbb{C} \setminus (I_1 \cup I_r)$ given by

$$q_j(z) = -\frac{\vartheta_1'(z_j/z)}{z\vartheta_1(z_j/z)} - \frac{z\vartheta_1'(z_j z)}{\vartheta_1(z_j z)},$$

maps the circles $\mathbb{S}_1$ and $\mathbb{S}_r$, onto two real intervals I_1 and I_r, respectively.

Characterization (see Ahlfords)

q_j is the unique (up to real additive constants) holomorphic map in $\mathbb{A}_r \setminus \{z_j\}$, which maps each boundary component of $\mathbb{A}_r$ onto a real interval and has a simple pole of residue 1 at z_j, $(q_j \sim \frac{1}{z - z_j})$.

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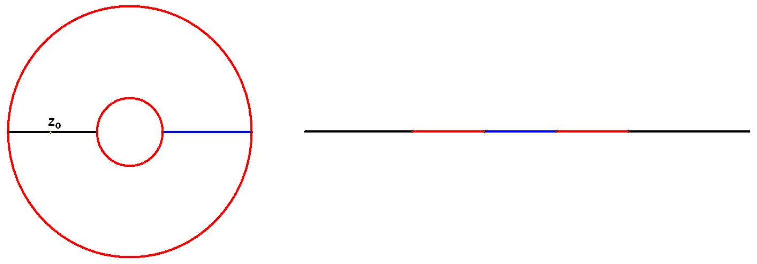
Canonical examples $(n = 2)$

Remark

If one takes $z_0 \in]-1, -r[$, then

$$q_0 :]z_0, -r[\cup \mathbb{S}_r \cup [r, 1] \cup \mathbb{S}_1 \cup]-1, z_0[\longrightarrow \mathbb{R}.$$



Canonical examples (n = 2)

Definition

Given $z_0, z_1, z_2 \in]-1, -r[$, we define $R : \mathbb{A}_r \setminus \{z_0\} \longrightarrow \mathbb{C}$ by

$$R(z) = aq_0(z) + b,$$

where a and b are real constants, determined by $R(z_1) = 1$, $R(z_2) = 0$ and such that $0 < R < 1$ on $\partial\mathbb{A}_r$.

From the above characterization one has

$$\frac{R'(z_1)}{R(z) - 1} = q_1(z) - c_1, \quad \frac{R'(z_2)}{R(z)} = q_2(z) - c_2,$$

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Proposition

If $z_0, z_1, z_2 \in]-1, -r[$ and $m \in \mathbb{R}$ satisfy

(C1) $m + c_1 z_1 - z_1 R'(z_1) = -z_2 R'(z_2)$,

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(C3) $z_1 z_2 r^{2(m+2)} = 1$.

Then the functions $g : \mathbb{A}_r \longrightarrow \mathbb{C}$ and $u : \mathbb{A}_r \setminus \{z_0\} \longrightarrow \mathbb{R}$ given by

$$g(z) = \sqrt{\frac{R(z)}{1 - R(z)} \frac{Q_1(z)}{Q_2(z)} z^{-2}}, \quad u(z) = \frac{1}{2} \log \left| \frac{Q_1(z)}{1 - R(z)} z^m \right|$$

with $Q_j(z) = \frac{\vartheta_1(z_j/z)}{\vartheta_1(z_j z)}$, $j = 1, 2$, are the data of a well-defined flat surface $\psi : \mathbb{A}_r \setminus \{z_0\} \to \mathbb{H}^3$, with $\psi(\mathbb{S}_1)$ and $\psi(\mathbb{S}_r)$ as isolated singularities.

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Notes of the Proof

- As z_j is a simple zero of Q_j, g is a holomorphic function and $u(z) - \frac{1}{2}\log|z - z_0|$ is a harmonic function in $\mathbb{A}_r$.
- From (C1), (C2) and $d\log Q_j(z) = \frac{z_j}{z}q_j(z)dz$ we get

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Canonical examples (n = 2)

Existence

For any $r \in]0,1[$ and $s \in]-1,0[$, there exist

$$m \in]-3,-2[$$

and

$$z_0, z_1, z_2 \in]-1,-r[,$$

$z_2 < z_0 < z_1$, which satisfy the conditions (C1), (C2), (C3), with

$$s = -z_2 c_2 = -z_2 q_2(z_0).$$

In particular, for $s = -1/2$, there is a solution with $m = -5/2$ and $z_0^2 = r = z_1 z_2$.

Canonical examples (n = 2)

Notes of the Proof

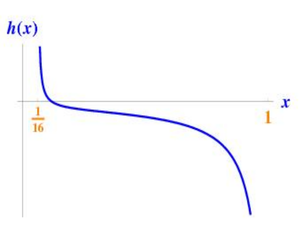
The conditions can be written as

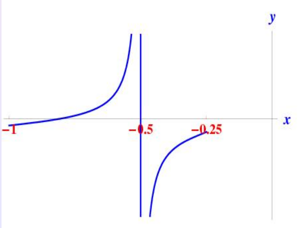
(C1) $m = 2h(r^{-2(m+2)}) - 1 - f_0(z_2)$,

(C2) $-2 = f_0(z_1) - f_0(z_2)$,

(C3) $z_1 z_2 = r^{-2(m+2)}$,

with $h(z) = \frac{z\vartheta_1'(z)}{\vartheta_1(z)}$, $f_0(z) = h(z/z_0) + h(zz_0)$.

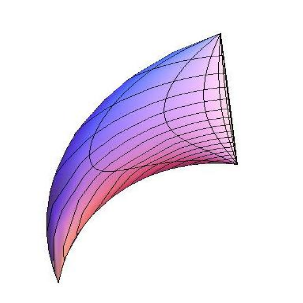




Canonical examples $(n = 2)$

Theorem

Each canonical example $\psi : \mathbb{A}_r \setminus \{z_0\} \longrightarrow \mathbb{H}^3$ is a complete flat embedding with two isolated singularities and one end.



Canonical examples $(n = 2)$

Notes of the Proof

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- Now
$$p(z) = \left(\frac{Q_1(z)}{1 - R(z)} z^m\right)^2 \left(\frac{F'(z)}{g'(z)} + F^2(z)\right)$$
verifies $|p(z)| = 1$ on $\partial\mathbb{A}_r \quad \Rightarrow \quad |p(z)| < 1$ on $\mathbb{A}_r$,
$$ds^2 = \exp(-4u)|\overline{dg} - p\, dg|^2 \geq \exp(-4u)|dg|^2(1 - |p|^2) = d\sigma^2$$
is positive definite, complete and $\psi(\mathbb{S}_1)$ and $\psi(\mathbb{S}_r)$ are their unique singularities.
- Only one embedded end $\psi(z_0)$ and ψ is an embedding.

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Characterizations

Theorem ($n = 1$)

The revolution examples are the unique complete flat embedding in $\mathbb{H}^3$ with only one isolated singularity and one end.

Theorem ($n = 2$)

Each complete flat embedding in $\mathbb{H}^3$ with only two isolated singularities and one end must be congruent to one of the canonical examples.

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Characterizations

Notes of the Proofs

The conditions on the boundary and in z_0 determine the above R, g and u.

In particular

$$du + \mathrm{i} *du = R \, \frac{dg}{g}$$

has a simple pole in z_0 and, up to isometries of $\mathbb{H}^3$, we can consider

1. The singularity and $\psi(z_0)$ in $\{0\} \times \mathbb{R}^+ \Rightarrow R$ is constant, since $\frac{dg}{g}$ has the pole in z_0.
2. The two singularities in $\{0\} \times \mathbb{R}^+ \Rightarrow R$ has the pole ...

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Remark

There are not compact embedded flat surfaces, with less than three isolated singularities, because R is constant only for the revolution examples.