The Asymptotic Geometry of Genus-g Helicoids

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Outline

Shapes of Complete Minimal Surfaces

Question

Can we classify complete, (properly) embedded minimal surface in $\mathbb{R}^3$ of finite topology?

Question

Can we classify the asymptotic geometry of these surfaces?

Theorem

(B.-Breiner, Collin, Meeks-Rosenberg,...). Let Σ be a complete, properly embedded minimal surface of finite topology then each end is asymptotic to a plane, a helicoid or half of a catenoid.

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Motivation

Why are we interested in such a question?

- Very classical problem, yet requires very sophisticated modern techniques.
- Physical motivations...
- Close relationship between understanding the shapes of these surfaces and compactness properties of sequences of embedded minimal surfaces.
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Minimal Surfaces

A brief introduction.

Let $\mathcal{E}(e, g)$ be the space of complete, embedded minimal surfaces in $\mathbb{R}^3$ of genus g and with e ends.

Definition

In $\mathbb{R}^3$ an immersed (oriented) surface Σ is minimal if:

- $\frac{d}{dt}|_{t=0} Area(\Sigma_t) = 0$ for all smooth compactly supported variations of $\Sigma = \Sigma_0$; or
- The mean curvature of Σ, H_Σ, vanishes identically ($\Rightarrow |A|^2 = -2K$); or
- The coordinate functions x_1, x_2, x_3 restrict to harmonic functions on Σ; or
- The Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^2$ is holomorphic.

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Constructing Examples

How do we get some examples of elements of $\mathcal{E}(e, g)$?

- Write down an explicit parameterization.
- Use the Weierstrass representation.
- Variational methods.
- Gluing constructions.

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Weierstrass Representation

Weierstrass representation: given triple (M, g, dh) of a Riemann surface, a meromorphic function and holomorphic one-form, can cook up a minimal immersion $F : M \to \mathbb{R}^3$ with $F^* dx_3 = \mathrm{Re}\, dh$ and with Gauss map determined by g.

$$F(p) = \mathrm{Re} \left(\int_{p_0}^{p} \left(\frac{1}{2} \left(g - \frac{1}{g} \right), \frac{1}{2i} \left(g + \frac{1}{g} \right), 1 \right) dh \right)$$

For F to be a well defined immersion on M need:

- $g\ dh$ and $g^{-1}\ dh$ to be holomorphic.
- $\int_{[\nu]} gdh - \overline{\int_{[\nu]} g^{-1} dh} = \int_{[\nu]} \mathrm{Re}\, dx_3 = 0 \ \ \forall [\nu] \in H^1(M)$.

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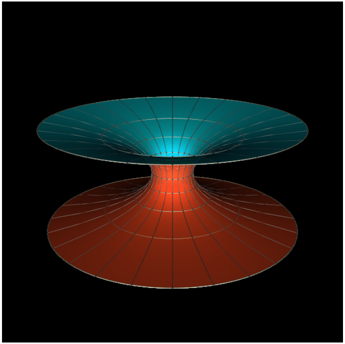
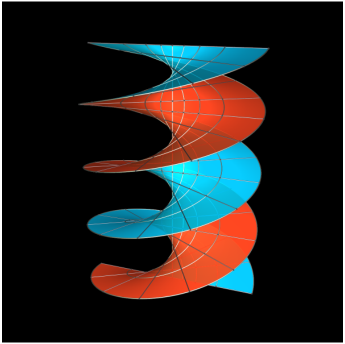
Classical Example: Catenoid



Catenoid (Euler, 1744).
In $\mathcal{E}(2, 0)$ and of finite total curvature.

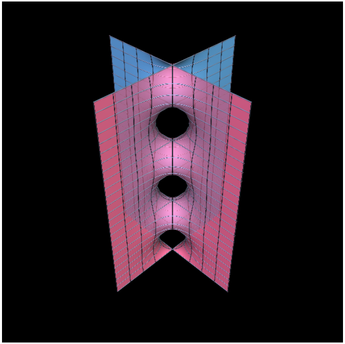
Images courtesy Matthias Weber, `http://www.indiana.edu/~minimal`.

Classical Example: Helicoid



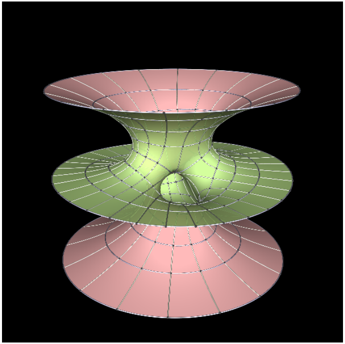
Helicoid (Meusnier, 1776)
In $\mathcal{E}(1, 0)$ and of infinite total curvature.

Classical Example: Scherk's Surface



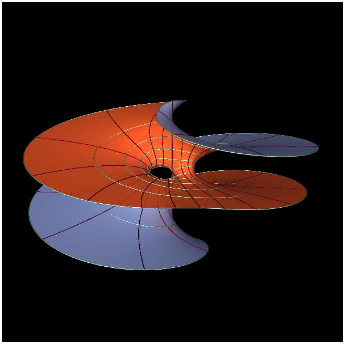
Scherk's Surface (Scherk, 1835)
In $\mathcal{E}(1, \infty)$ and of infinite total curvature.

Modern Example: Costa Surface



Costa Surface (Costa, '83).
Proven embedded by Hoffman and Meeks in '84.
In $\mathcal{E}(3, 1)$ and of finite total curvature.

Modern Example: Genus-One Helicoid



Genus-One Helicoid (Hoffman, Karcher and Wei, '93).
Proven embedded by Hoffman, Weber and Wolf in '04.
In $\mathcal{E}(1, 1)$ and of infinite total curvature.

Strategy for Classifiying the Ends

The close interaction between complex analysis and the geometry of a minimal surfaces gives a general procedure for trying to classify the asymptotic geometry:

- Step 1: Get some weak control on the asymptotic geometry.
- Step 2: Use this to bound the Weierstrass data of the end.
- Step 3: As the Weierstrass data holomorphic, get finer understanding.
- Step 4: Use Weierstrass representation together with embeddedness for further refinement.

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Classical Result

A classical result of Osserman provides a good example of this.

Theorem

(Huber '58, Osserman, '64) Let Σ be a complete minimal surface with finite total curvature in $\mathbb{R}^3$ then Σ is conformal to a finitely punctured Riemann surface and its Gauss map extends holomorphically to each end. Thus, if the surface is embedded each end is asymptotic to a plane or half a catenoid.

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Complete Surfaces with 2 Ends

For $\mathcal{E}(e, g)$ when $e \geq 2$ barrier constructions imply finite total curvature.

Theorem

(Meeks and Rosenberg, '93) If $\Sigma \in \mathcal{E}(e, g)$ and $e \geq 2$ then Σ is conformal to a punctured compact Riemann surface and at least $e - 2$ of the ends are asymptotic to a plane or a catenoid.

Theorem

(Collin, '97) If $\Sigma \in \mathcal{E}(e, g)$ and $e \geq 2$ then Σ has finite total curvature and so is conformal to a punctured compact Riemann surface and each end is asymptotic to a plane or a catenoid.

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Surfaces with One End

- Complete minimal surfaces with one end must have infinite total curvature or be the plane.
- Thus, Huber and Osserman's result cannot be applied and so no a priori knowledge about conformal type of the end. Some results with additional assumptions on the asymptotic behavior. (cf. Hoffman-McCuan, Hauswirth-Pérez-Romon)
- General results only achieved by applying a new theory developed by Colding and Minicozzi. Their work has (among other things) led to the complete classification of the asymptotic geometry.

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Conformal and Asymptotic Properties of $\mathcal{E}(1, g)$

Elements of $\mathcal{E}(1, 0)$ are completely classified.

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Incomplete Surfaces

Interesting to consider each end individually (i.e. as a surface with compact boundary and one end).

One expects similar arguments except:

- Weak asymptotics less straightforward.
- Subtleties involving the flux arise.
- Good example is recent work of Meeks and Pérez.

Indeed, they consider ends with infinite total curvature. If the flux is zero around boundary the surface is asymptotic to a helicoid. If non-zero it is asymptotic to a certain family they construct.

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Shapes of Embedded Disks

Let $0 \in \Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$.

Theorem

(Colding and Minicozzi, '04) There exist constants $C, \Omega > 1$ so if Σ is as above and $|A|^2(0) > CR^{-2}$ then the component of $B_{R/\Omega} \cap \Sigma$ containing 0 is the union of two multi-valued graphs that spiral together.

The previous theorem can be interpreted as saying Σ looks (away from the boundary) like a distorted helicoid. This distortion can be quite great, and so in principle description is *qualitative*.

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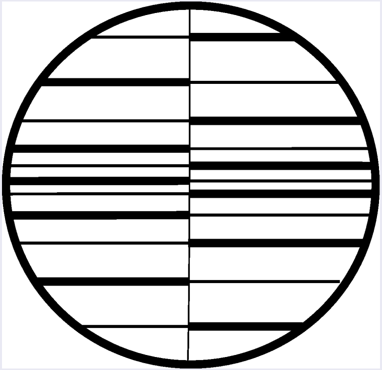
Colding and Minicozzi's Examples

Tight Spiraling

Colding and Minicozzi also give an example of a sequence of disks $0 \in \Sigma_i$ with $\partial\Sigma_i \subset \partial B_1$ that have uniformly bounded curvature away from the origin but have curvature blowing up at the origin.

Image

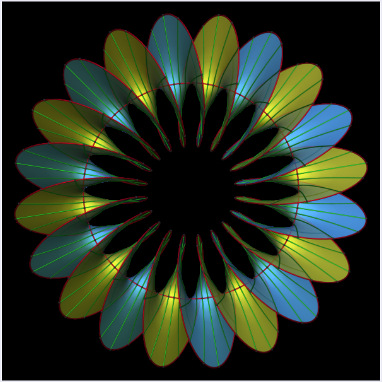


Bent Helicoids

Bent Helicoids

Meeks and Weber have constructed examples of “bent” helicoids, where the multi-valued graphs have axis an arbitrary $C^{1,1}$ curve.

Image



Sketch of Colding and Minicozzi's Argument

- First focus on points of large curvature:

Definition

A pair $(y, s) \in \Sigma \times \mathbb{R}^+$ is a (C) blow-up pair if $\sup_{B_s(y)} |A|^2 \leq 4|A|^2(y) = 4C^2 s^{-2}$.

- Suppose (y, s) is such a pair in Σ, far from $\partial\Sigma$, then near y a small multi-valued graph, Σ_0, forms on the scale s
- Σ_0 can be extended, as a multi-valued graph, in Σ to $\partial\Sigma$.
- Points of Σ "between the sheets" of Σ_0 form a second multi-valued graph.
- The existence of these graphs implies existence of blow-up pairs above and below y.
- Process can be iterated and rest of Σ can be filled in.

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- Σ_0 can be extended, as a multi-valued graph, in Σ to $\partial\Sigma$.
- Points of Σ "between the sheets" of Σ_0 form a second multi-valued graph.
- The existence of these graphs implies existence of blow-up pairs above and below y.
- Process can be iterated and rest of Σ can be filled in.

Sketch of Colding and Minicozzi's Argument

- First focus on points of large curvature:

Definition

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Some Applications

This description of disks has some powerful consequences:

- Curvature bounds for embedded minimal disks that are close to, but on one side of, a plane. Effective version of the strong half-space theorem.
- Compactness theory for embedded minimal disks that requires no curvature or area bounds.
- Chord-arc bounds for embedded minimal disks, i.e. uniform relationship between intrinsic and extrinsic length. Used to settle Calabi-Yau conjecture for embedded minimal disks.

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Description of $\mathcal{E}(1, g)$

We now discuss the proof of:

Theorem

Every element of $\mathcal{E}(1, g)$ is conformal to a once-punctured compact genus g Riemann surface and is either a plane or is asymptotic to a helicoid.

For simplicity we will focus on the case $g = 0$ and indicate how one generalizes.

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Strategy

Recall our strategy for understanding the ends:

- Step 1: Get some weak control on the asymptotic geometry.
- Step 2: Use this to bound the Weierstrass data of the end.
- Step 3: As the Weierstrass data holomorphic, get finer understanding.
- Step 4: Use Weierstrass representation together with embeddedness for further refinement.

Step 1: Initial Decomposition

Due to distortions, Colding and Minicozzi's description alone does not suffice.

However, for complete Σ, it can be refined:

Theorem

Let Σ be a non-flat, complete, properly embedded minimal disk. There exist disjoint sets $\mathcal{R}_A$ and $\mathcal{R}_S$ with $\Sigma = \mathcal{R}_A \cup \mathcal{R}_S$ and an $\epsilon_0 > 0$ so, after a rotation in $\mathbb{R}^3$, $\mathcal{R}_S$ is the union of two strictly spiraling multi-valued graphs and, on $\mathcal{R}_A$, $|\mathbf{n} \cdot \mathbf{e}_3| \leq \epsilon_0$.

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Proof of the Decomposition

Proof.

- Multi-valued minimal graphs over unbounded annuli eventually strictly spiral.
- Colding and Minicozzi's work allows one to construct the strictly spiraling region $\mathcal{R}_S$ in Σ.
- Their work also gives understanding of $|A|$ in $\mathcal{R}_A$.
- Harmonicity of coordinate functions $\Rightarrow \nabla_\Sigma x_3 \neq 0$.
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Step 2: Conformal Structure of Σ

The decomposition determines the conformal type of Σ:

- $z = x_3 + ix_3^*$ is a holomorphic coordinate on Σ. We claim it is onto, that is Σ is "conformally" once-punctured (i.e. have Huber's result).
- $\nabla_\Sigma x_3 \neq 0 \iff \mathbf{n} : \Sigma \to \mathbb{S}^2 \backslash (0, 0, \pm 1) \Rightarrow$ stereographically projecting gives a holomorphic map $g : \Sigma \to \mathbb{C} \backslash \{0\}$.
- $\exists h$ so $e^h = g$. Note, $|\mathrm{Re}\, h| \leq \gamma_0$ on $\mathcal{R}_A$.
- Idea: know h "almost" surjective whereas know z injective.
- Indeed: h injective on subset Ω of $\mathcal{R}_S$ and maps subset *onto* two closed half-planes.

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Step 3 and 4: Concluding Uniqueness

We now understand conformal type of the end.

- As h is injective on Ω have: $h \circ z^{-1} : \mathbb{C} \to \mathbb{C}$ linear, i.e. $g(p) = e^{\alpha z(p)+\beta}$, $\alpha, \beta \in \mathbb{C}$.
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Open Questions

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Is $\mathcal{E}(1, g)$ non-empty for $g > 1$?

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Can we classify $\mathcal{E}(1, 1)$? Does it have a unique element (modulo rigid motions and homotheties)?

Can show any element of $\mathcal{E}(1, 1)$ admits an orientation preserving involutive symmetry (induced by rotation about a line).

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