GROWTH PROPERTIES OF SOLUTIONS TO THE MINIMAL SURFACE EQUATION

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The Dirichlet Problem for the Minimal Surface Equation

$$(\bigstar) \qquad \mathrm{div}\frac{\vec{\nabla} u}{\sqrt{1+|\vec{\nabla} u|^2}} = 0 \quad \text{in } D, \qquad u = \Phi \quad \text{on } \partial D.$$

1. If D is bounded and convex, then $(\bigstar)$ has a unique solution. If D is not convex, there are always boundary functions Φ for which there is no solution. In the convex case, the graph of u gives the surface which minimizes area amongst all surfaces with boundary function Φ.

2. Solutions to $(\bigstar)$ have a (very) strong maximum principle.

3. Solutions to $(\bigstar)$ are real analytic.

4. The surface has nonpositive curvature at each point.

5. If D is the plane, then u is affine.

Nitsche's Theorem

We consider the problem (★) with vanishing boundary values.

$$\operatorname{div} \frac{\vec{\nabla} u}{\sqrt{1+|\vec{\nabla} u|^2}} = 0 \qquad u > 0 \qquad \text{in } D,$$

(★★)

$$u = 0 \qquad \text{on } \partial D.$$

By the maximum principle, D must be unbounded.

Theorem. (Nitsche 1965) No solutions if D is contained in a sector of opening less than π.

So minimal graphs coming from (★★) "take up a lot of room".

The Asymptotic Angle

If D is a domain, then define

$$\Theta(r) = \mathrm{meas}_\theta \left(D \cap \{|z| = r\}\right)$$

and the *asymptotic angle*

$$\beta = \limsup_{r \to \infty} \Theta(r).$$

Based on Nitsche's theorem we have

Conjecture

There are no solutions to (★★) with $\beta < \pi$.

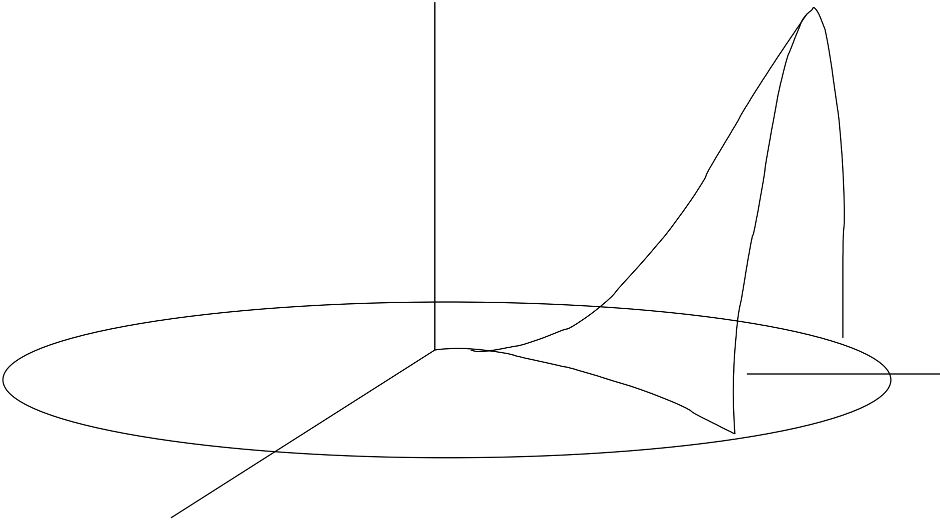
In classical potential theory, the asymptotic angle and growth rates of harmonic and subharmonic functions are related.
The *order* of $u(z)$ is

$$\alpha = \limsup_{z \to \infty,\ z \in D} \frac{\log |u(z)|}{\log |z|}.$$

Theorem If u is a nontrivial solution to (★★), D is bounded by a Jordan arc with asymptotic angle $\beta \geq \pi$, then

$$\alpha \geq \pi/\beta.$$

Growth Properties of Solutions to (★★)



Conjecture

If $u(z)$ is a solution to ($\bigstar\bigstar$), then for $z \in D$,

$$|u(z)| < Ke^{K|z|}.$$

Example. The portion of the catenoid over the right half plane where $u > 0$

$$u(x,y) = \left(\sqrt{\cosh^2 Cx - C^2y^2} - 1\right)/C$$

Theorem If u is a solution to ($\bigstar\bigstar$), D lies in a halfplane and is bounded by a Jordan arc, then for $z \in D$,

$$|u(z)| < Ke^{K|z|}.$$

Lower Bounds

Theorem If u is a nontrivial solution to ($\bigstar\bigstar$), D lies in a halfplane and is bounded by a Jordan arc, then for $z \in D$,

$$\max_{|z|=r} |u(z)| > Kr$$

Conjecture

If u is a nontrivial solution to ($\bigstar\bigstar$) and D is simply connected, then for $z \in D$,

$$\max_{|z|=r} |u(z)| > Kr^{1/2}$$

So far we have represented S nonparametrically by $(x, y, u(x, y))$ where u is as in $(\bigstar)$. We may also use the *Weierstrass representation* to represent S locally (and globally if D is simply connected) parametrically in conformal coordinates $(x_1(\zeta), x_2(\zeta), x_3(\zeta))$.

Notations

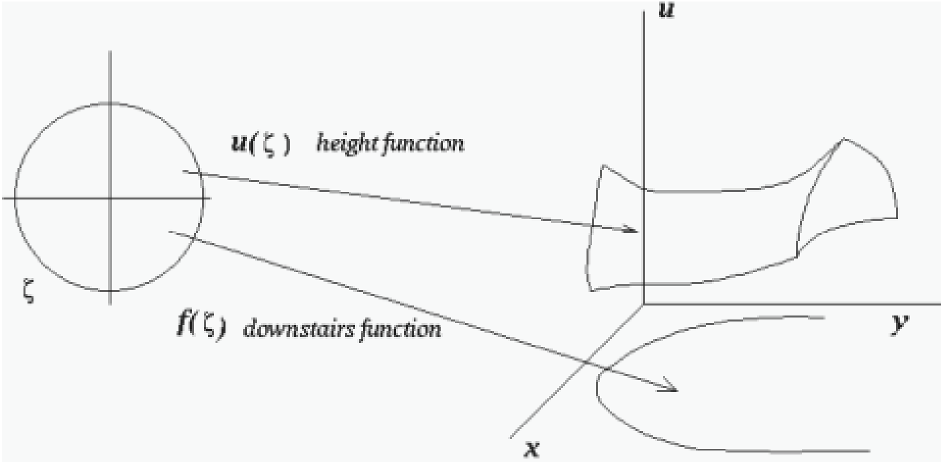
$$f(\zeta) = x_1(\zeta) + ix_2(\zeta) = h(\zeta) + \overline{g(\zeta)} = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \quad \zeta \in U$$

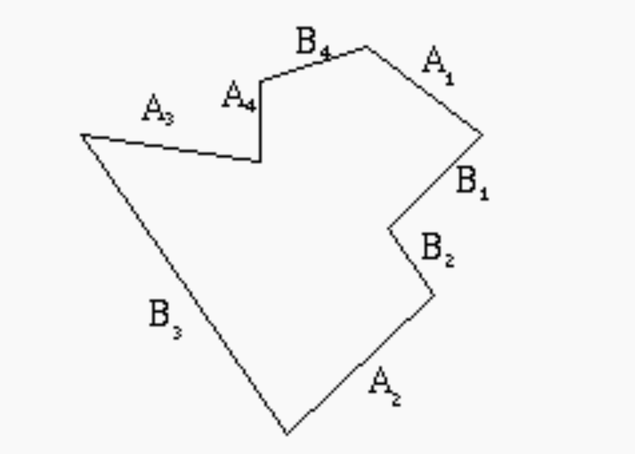
$$u(x_1(\zeta), x_2(\zeta)) = \Im m\, F(\zeta) = \Im m\, 2 \int \sqrt{h'(\zeta) g'(\zeta)}\, d\zeta$$

$$ds = (|h'(\zeta)| + |g'(\zeta)|)|d\zeta|$$

$$a(\zeta) = \overline{f_{\overline{\zeta}}(\zeta)}/f_\zeta(\zeta) = g'(\zeta)/h'(\zeta) = -1/G^2(\zeta).$$

$$K(\zeta) = \frac{-|a'(\zeta)|^2}{|h'(\zeta)g'(\zeta)|(1 + |a(\zeta)|)^4}.$$


u
u(ζ) height function
ζ
f(ζ) downstairs function
y
x



JS surfaces: $u = +\infty$ on int A_j u= $-\infty$ on int B_j.

Scherk's surface
$u(x_1, x_2) = \log(\cos x_1 / \cos x_2) \quad -\pi/2 < x_1,\ x_2 < \pi/2$

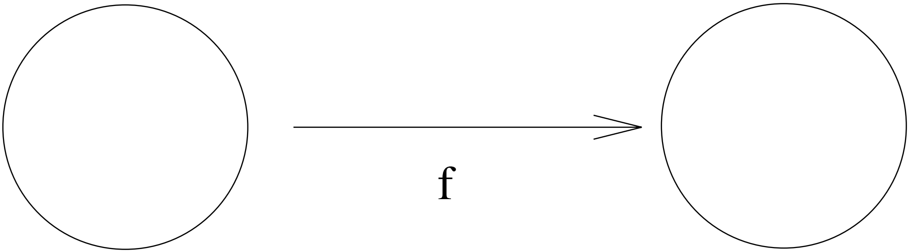
Jenkins and Serrin gave necessary and sufficient conditions for the existence of JS surfaces. In particular, $+\infty,\ -\infty$ must alternate at convex corners.

The downstairs function $f(\zeta)$ is the Poisson integral of a step function, namely the vertices of the polygon.

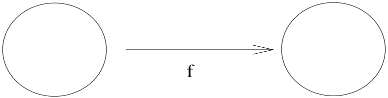
If the height function has n sign changes, $G(\zeta) = c/B(\zeta)$ where $B(\zeta)$ is a Blaschke product of order $(n-2)/2$.

E. Heinz 1950's. If $D = U$, then $|K(0)| \leq K_0$. Normalizing the f corresponding to the surface by $f(0) = 0$,

$$|K(0)| \leq \frac{4}{|a_1|^2 + |a_{-1}|^2}.$$



Function Theoretic Estimates



Schwarz Lemma

f harmonic (not necessarily univalent) $U \to U$, $f(0) = 0$, then $|f(z)| \leq (4/\pi)\tan^{-1}|z|$.

$f(U) = U$ and f 1-1,
Duren and Schober (1987,1989):

$$|a_0| < 1 \quad |a_1 \leq 1, \quad |a_n| < 1/n \quad n \geq 2,$$

$$|a_n| < \frac{n+1}{n\pi}\sin(\frac{\pi}{n+1}) \quad n < 0.$$

Hall (1985): $|a_1|^2 + (3\sqrt{3}/\pi)|a_0|^2 + |a_{-1}|^2 > 27/(4\pi^2)$
W. (1998) : $|a_0| + |a_1| > 2/\pi$

$$f(z) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \quad \zeta \in U$$

$f \in S_H^o$ if f is univalent, and normalized so that $a_0 = a_{-1} = 0$ and $a_1 = 1$.

Harmonic Koebe Function

$$K(z) = \Re e \frac{z + (1/3)z^3}{(1-z)^3} + i \Im m \frac{z}{(1-z)^2}$$

Harmonic Bieberbach Conjecture for S_H^o

$$|a_n| \leq \frac{1}{6}(2n+1)(n+1)$$

$$|a_{-n}| \leq \frac{1}{6}(2n-1)(n-1)$$

Suppose D unbounded and simply connected with ∂D a piecewise differentiable Jordan arc not containing the origin. Then D will be a *spiraling domain* and its graph S from (★★) a *spiraling minimal graph*, if ∂D contains a subarc β tending to ∞ on which, for a branch of arg z on β, we have

$$\arg_{z\in\beta} z \uparrow +\infty \quad \text{as } z \to \infty.$$

Question

Do spiraling minimal graphs exist?

Answer

Yes.

Question

Are there restrictions on spiraling?

Answer

Yes.

Theorem. Let D be a spiraling domain with β as above and suppose that u is nontrivial and satisfies (★★). Then there is a constant τ_0 such that if the limit

$$\tau(\beta) = \lim_{z \to \infty \ z \in \beta} \frac{\arg z}{\log |z|},$$

exists, then $\tau \leq \tau_0$.