# Lagrangian (and affine) immersions for which suitable tensors are isotropic

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# Plan

Lagrangian submanifolds
- Introduction
- First results
- Idea of proof
- Possible generalisations

Affine differential geometry

Other geometric tensors

The indefinite case

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The notion of a submanifold with isotropic second fundamental form was first introduced by O'Neill. Namely, if

$$< h(X(p), X(p)), h(X(p), X(p)) >= \lambda(p) < X(p), X(p) >^2,$$

for any $X(p) \in T_pM$, we say that $M$ has isotropic second fundamental form. If $\lambda$ is independent of the point $p$, the submanifold is called constant isotropic.

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For Lagrangian submanifolds, the first result about isotropic submanifolds was obtained by Naitoh, in his study of submanifolds with parallel second fundamental form. Later such submanifolds were studied by Montiel and Urbano (1988) who showed the following results:

**Theorem**

*Let $M^n$ be a Lagrangian submanifold of $\mathbb{C}P^n(4)$. If $M$ is constant isotropic then $M$ has parallel second fundamental form.*

**Theorem**

*Let $M^n$, $n > 2$, be a minimal Lagrangian submanifold of $\mathbb{C}P^n(4)$. Assume that $M$ is not totally geodesic. If $M$ has isotropic second fundamental form then $M$ is constant isotropic and either $n = 5, 8, 14$ or $25$.*

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Let $p \in M$. As

$$UM_p = \{v \in T_pM | < v, v >= 1\},$$

is compact, we can choose $e_1$ such that

$$f : UM_p \to \mathbb{R} : v \mapsto < h(v,v), Jv >,$$

attains an absolute maximum for $v = e_1$. This implies that $e_1$ is an eigenvector of the symmetric operator $A_{Je_1}$.

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Using then the isotropy condition it follows that

$$T_pM = T_0 \oplus T_1 \oplus T_2,$$

where

1. $T_0$ is spanned by $e_1$,
2. $T_1$ is the eigenspace of $A_{Je_1}$ with eigenvalue $-\lambda$
3. $T_2$ is the eigenspace of $A_{Je_1}$ with eigenvalue $\frac{\lambda}{2}$.

The isotropy condition implies that

$$h(v,w) = -\lambda < v,w > Je_1, \qquad v,w \in T_1$$

whereas the fact that $f$ attains an absolute maximum in $e_1$ implies that

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$$A_{Je_2} : T_2 \to T_2,$$

with

$$< A_{Je_2} v, A_{Je_2} v >= \tfrac{3}{4}\lambda^2$$

This shows that we can decompose $T_2 = V_2^+ \oplus V_2^-$, which are respectively the eigenspaces with eigenvalue $\frac{\sqrt{3}}{2}\lambda$ and $-\frac{\sqrt{3}}{2}\lambda$. In view of the dimension, both of the above spaces must have the same dimension which also must be equal to the dimension of $T_1$.

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From before,

$$h(x^+, x^-) \in T_1$$

and orthogonal to $Je_2$. Moreover, the isotropy condition (and the fact that the metric is positive definite) implies that

$$h(x^+, x^+) =< x^+, x^+ > \lambda(\tfrac{1}{2}Je_1 + \tfrac{\sqrt{3}}{2}Je_2)$$

and

$$h(x^-, x^-) =< x^-, x^- > \lambda(\tfrac{1}{2}Je_1 - \tfrac{\sqrt{3}}{2}Je_2)$$

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$$T(X, Y, Z, W) =< \nabla h)(X, Y, Z), JW >$$

   is isotropic

2. Study Lagrangian submanifolds for which

$$T(X, Y, Z, W, U, V) =< \nabla h)(X, Y, Z), (\nabla h)(W, U, V) >$$

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3. Submanifolds with isotropic fundamental tensors in affine differential geometry
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# Affine differential geometry

In terms of the difference tensor $K$, the affine metric and the affine shape operator $S$, the basic equations which are given by

$$\hat{R}(X,Y)Z = \tfrac{1}{2}(h(Y,Z)SX + h(SY,Z)X - h(X,Z)SY - h(SX,Z)Y) - [K_X, K_Y]Z \quad (1)$$

$$(\hat{\nabla}K)(X,Y,Z) - (\hat{\nabla}K)(Y,X,Z) = \tfrac{1}{2}(h(Y,Z)SX - h(SY,Z)X - h(X,Z)SY + h(SX,Z)Y) \quad (2)$$

$$(\hat{\nabla}_X S)(Y) + K(X,SY) = (\hat{\nabla}_Y S)(X) + K(Y,SX)$$

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# Theorem (Birembaux-Djoric)

*Let $n \geq 3$ and $M$ be an $n$-dimensional affine sphere in $\mathbb{R}^{n+1}$ which is $\lambda$-isotropic. Then $M$ is a constant isotropic hyperbolic affine sphere and $M$ is affine equivalent with a canonical immersion of one of the following symmetric spaces:*

- $SL(3,\mathbb{R})/SO(3)$;
- $SL(3,\mathbb{C})/SU(3)$;
- $SU^*(6)/Sp(3)$;
- $E_6(-26)/F_4$.

What happens if $M$ is not an affine sphere?

Of course the dimension can only be 5, 8, 14 or 26.

For dimension 5 a complete answer is almost known (joint work with Birembaux and Djoric).

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## Idea of the proof

The first step is to derive the isotropy condition, at a point $p$ of $M$, and determine relations between the $< (\hat{\nabla} K)(e_i, e_j, e_k), e_\ell >$ explicitly. Next we use the Codazzi equation for $\hat{\nabla} K$. Combining the above equations it is possible to determine explicitly

1. the components of $< (\hat{\nabla} K)(e_i, e_j, e_k), e_\ell >$
2. the components of $< S(e_i), e_j >$

in terms of the components of the gradient of $\lambda$ and the mean curvature.

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Next we use the fact that the choice of the frame is not unique. In the 5-dimensional case, there exists an $SO(3)$-family of possible frames. We can then choose are $e_1$ that, amongst all possible $e_1$'s, the component of the gradient of $\lambda$ in the direction of $e_1$ is the largest one possible. Doing so we find that $e_1$ is an eigenvector of the shape operator. As the eigenspace corresponding to this eigenvalue is 1-dimensional, it follows that our frame can be extended differentiably.

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Using the above choice of $e_1$, it follows that the gradient of $\lambda$ has no component in the direction of $T_2$. Introducing the connection coefficients and using the coefficients of $< (\hat{\nabla} K)(e_i, e_j, e_k), e_\ell >$ it follows that there are two cases to be considered, namely:

Case 1: $gradient(\lambda) \in T_0$,

Case 2: $gradient(\lambda) \in T_0 \oplus T_1$,

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## The first case

We obtain the following immersion:

$$F = (e^{-\sqrt{3}x} + 3y^2 e^{\sqrt{3}x} + 6w^2, ye^{\sqrt{3}x}\alpha_1(u) + 2\sqrt{3}vw\alpha_2(u), \\ e^{\sqrt{3}x}\alpha_1(u) + 6v^2\alpha_2(u), w\alpha_2(u), v\alpha_2(u), \alpha_2(u)) \qquad (3)$$

where

$$\alpha_1(u) = 1/c(2(1 - e^{cu})^{\frac{3}{2}})e^{-cu} \\ \alpha_2(u) = 2/ce^{-cu}.$$

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$$\alpha_2(u) = 2/ce^{-cu}.$$

# The second case

We first look at the 2-dimensional distribution determined by $T_0$ and the gradient of $\lambda$.

It turns out that this distribution is totally geodesic (therefore integrable) and that its leaves are affine surfaces in a 3-dimensional space $\mathbb{R}^3$.

Moreover, the surface admits an isothermal coordinate such that the difference tensor satisfies $K(\partial, \partial) = \bar{\partial}$, together with an additional equation on the metric.

Such surfaces can be classified.

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# Other geometric tensors

Theorem

*(Li,-) Let* $M^n \to \mathbb{C}P^n(4)$ *be a Lagrangian submanifold. If the tensor*

$$T(X, Y, Z, W) = < (\nabla h)(X, Y, Z), JW >,$$

*is isotropic. Then, either* $M$ *has parallel second fundamental form or* $M$ *is congruent to the Whitney sphere.*

## Theorem

*(Li,-,Wang) Let $M^3 \to \mathbb{C}P^3(4)$ be a Lagrangian submanifold. If the tensor*

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*is isotropic. Then, either $M$ has parallel second fundamental form, $M$ is congruent to the Whitney sphere or $M$ has isotropic second fundamental form (H-umbilic)*

# The indefinite case

This is a work in progress with F. Dillen (Leuven) and H. Li and X. Wang (Tsinghua University). Given: an indefinite minimal Lagrangian submanifold such that for any tangent vector$x \in T_pM$

$$< h(x,x), h(x,x) >= \lambda(p) < x,x >^2,$$

where $\lambda$ is a non vanishing function. The above notion was introduced by P.M. Chacon and G.A. Lobos.

Our aim is to determine the possible dimensions, the possible expressions of the second fundamental form and also the possible immersions.

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$$< h(x,x), h(x,x) >= \lambda(p) < x,x >^2,$$

where $\lambda$ is a non vanishing function. The above notion was introduced by P.M. Chacon and G.A. Lobos.

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Our aim is to determine the possible dimensions, the possible expressions of the second fundamental form and also the possible immersions.

We consider three cases:

1. For any null vector $v$, $Jv$ and $h(v,v)$ are linearly dependent
2. For any null vector $v$, $h(v,v)$ is orthogonal to $Jv$. However there exists a null vector $u$ such that $h(u,u)$ and $Ju$ are linearly independent
3. There exists a null vector $v$ such that $< h(v,v), Jv > \neq 0$.

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Writing $K(x, y) = -Jh(x, y)$ and linearizing the isotropy condition, we get

$$\sigma(K(K(x, y), z)) = \sigma(< x, y > z),$$

where $\sigma$ denotes cyclic permutation.
We now rescale $v$ such that $< h(, v, v), Jv >= -4\lambda^2$ and define

$$u_1 = v$$
$$u_2 = K(v, v).$$

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$$K(u_1, u_1) = u_2$$
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Showing that the space spanned by $u_1$ and $u_2$ is invariant under $K$. We then take the orthogonal complement of $\{u_1, u_2\}$ and we look at the Jordan form of $K_{u_1}$

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By a similar argument as before, it follows that $K_{u_1}$ is diagonalisable over the complex numbers and that the possible eigenvalues are

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$$\lambda(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2})$$
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Writing now

$$e_1 = \frac{1}{d}(2\lambda f_1 - f_2)$$
$$e_2 = \frac{1}{d}(2\lambda f_1 + f_2)$$

where $d$ is chosen such that $e_1$ has length $\pm 1$
we get the same decomposition of the tangent space as in the positive definite case with the exception that $V_2^+$ (and $V_2^-$) are complexified eigenspaces with eigenvalues $\pm i\frac{\sqrt{3}}{2}$
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Results so far:

$p = 1$, $n = 5$: 1 possible second fundamental form leading to 1 example

$p = 2$, $n = 8$: 2 possible second fundamental forms leading to 2 examples (with different signatures for the metric)

$p = 3$, $n = 11$: no examples

Lorentzian case (minimality assumption not necessary): only H-umbilical examples can occur. These can be all classified.

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