Principles of Einstein–Finsler Gravity

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Review Lecture

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Differential & Finsler Geometry, Iaşi, Romania

Research group "Geometry & Applications in Physics"

100 years traditions on math & applications; supervision/ collaborations by/with **D. Hilbert, T. Levi–Civita** and **E. Cartan** of PhD of prominent members of Romanian Academy.

- E. Cartan visit at Iaşi in 1931 induced 80 years of research on Finsler/integral geometry etc, "isolation" after 1944; "Japanese–Finsler geometry orientation" after 1968
- Alexandru **Myller** (1879–1965), PhD–1906: D. Hilbert (chair/adviser) and F. Klein, H. Minkowski (commission).
- Gheorghe **Vrănceanu** (1900–1979), PhD-1924, from Levi–Civita, commission head: Volterra; 1927-28, Rockefeller scholarship for France, E. Cartan, and USA at Harvard & Princeton (Morse, Birkhoff, Veblen)

Differential & Finsler Geometry, Iaşi, Romania
(prolongation)

- Mendel **Haimovici** (1906–1973); PhD-1933- Levi–Civita.
- Radu **Miron** (1927 -); 28 monogr., 240 rev. MathSciNet Lagrange–Finsler, Hamilton–Cartan & higher order, applications to mechanics and relativity etc.
- Iaşi team and "Romanian Finsler diaspora": M. Anastasiei, D. Bucătaru and M. Crâsmâreanu (Iaşi); A.Bejancu(Kuwait);D.Hrimiuc(Canada);V.Sabau(Japan); S. Vacaru (Cernâuţi/Chernivtsy, Chişinâu/ Kishinev, Tomsk, Dubna, Moscow, Kyiv, Bucharest–Măgurele, Lisbon, Madrid, Toronto, Iaşi)

Outline

Goals

- Finsler modifications of GR derived for QG theories; Geometric models for quantum contributions and LV
- Nonholonomic evolutions of (pseudo) Riemannian geometries into Lagrange–Finsler ones
- Canonical models for Einstein–Finsler gravity (EFG); principles and axioms
- Physical implications in EFG: Finsler branes, locally anisotropic cosmology & astrophysics

Reviews and new results:
S. Vacaru (in CQG, PLB, IJGMMP, JMP, JGP, IJTP)
arXiv: 1008.4912; 1004.3007; 1003.0044;
0909.3949; 0907.4278

Motivation: nonlinear disps; QG & LV, cosmology

1. Deforms in Minkovski s-t: $E^2 = p^2 c^2 + m_0^2 c^4 + \varphi(E, p; \mu; M_P)$
$E \sim \frac{\partial}{\partial t}, p_{\widehat{i}} \sim \frac{\partial}{\partial x^{\widehat{i}}}, \omega = \frac{\partial \phi}{\partial t}\ k_{\widehat{i}} = \frac{\partial \phi}{\partial x^{\widehat{i}}}, \omega^2 = c_s^2 k^2 + c_s^2 (\frac{\hbar}{2 m_0 c_s})^2 k^4 + ...$
effective c_s, $(x^1 = ct, x^2, x^3, x^4); \widehat{i}, \widehat{j}... = 2, 3, 4;$

$$\omega^2 = c^2 [g_{\widehat{i}\widehat{j}} k^{\widehat{i}} k^{\widehat{j}}]^2 (1 - q_{\widehat{i}_1 \widehat{i}_2 \ldots \widehat{i}_{2r}} y^{\widehat{i}_1} ... y^{\widehat{i}_{2r}} / r [g_{\widehat{i}\widehat{j}} k^{\widehat{i}} k^{\widehat{j}}]^{2r})$$

light velocity in "media/ether" $c^2 = g_{\widehat{i}\widehat{j}}(x^i) y^{\widehat{i}} y^{\widehat{j}} / \tau^2 \rightarrow \check{F}^2(y^{\widehat{j}})/\tau^2$
fundamental Finsler function $F(x^i, \beta y^j) = \beta F(x^i, y^j), \beta > 0,$

$$ds^2 = F^2 \approx -(cdt)^2 + g_{\widehat{i}\widehat{j}}(x^k) y^{\widehat{i}} y^{\widehat{j}} [1 + \frac{1}{r} \frac{q_{\widehat{i}_1 \widehat{i}_2 \ldots \widehat{i}_{2r}}(x^k) y^{\widehat{i}_1} \ldots y^{\widehat{i}_{2r}}}{(g_{\widehat{i}\widehat{j}}(x^k) y^{\widehat{i}} y^{\widehat{j}})^r}] + O(q^2)$$

Finsler "metrics", velocities on TV, ${}^F g_{ij}(x^i, y^j) = \frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j}$
2. Nonholonomic Ricci flows and mutual transforms of Riemann–Finsler geometries.
3. Exact solutions & modified cosmology with generic off–diagonal metrics and local anisotropy.

Einstein–Finsler Gravity (EFG)

Statement I: A (pseudo) Finsler metric, ${}^{F}g_{ij}(x^k, y^a)$, DOES NOT define completely a geometric model (not Riemannian !)

Statement II: A model of Finsler geometry is defined on TV by THREE fundamental geometric objects induced by $F(x, y)$:

1. N–connection, $N_i^a(x, y)$, splitting ${}^{F}\mathbf{N} : TTV = hTV \oplus vTV$ canonically, Euler–Lagrange for $L = F^2$ are semi–sprays,
2. d–connection, N–adapted linear connect. ${}^{F}\mathbf{D} = (hD, vD)$, preferred/ canonically induced by ${}^{F}g_{ij}$ and N_i^a
3. d–metric, ${}^{F}\mathbf{g} = h\mathbf{g} \oplus v\mathbf{g}$

2 classes: a) nonmetricity, ${}^{F}\mathbf{Q} := {}^{F}\mathbf{D}\,{}^{F}\mathbf{g}$, Chern d–conn., ${}^{Ch}\mathbf{D}$
b) metricity, ${}^{F}\mathbf{Q} = 0$, Cartan d–conn., ${}^{Cart}\mathbf{D}$

Levi–Civita ${}^{F}\nabla$ is NOT adapted to nonholonomic ${}^{F}\mathbf{N}$.

$\exists$ induced by ${}^{F}\mathbf{g}$: torsion ${}^{F}\mathbf{T}$, and/or ${}^{F}\mathbf{Q}$(not Riemann-Cartan)

Einstein–Finsler spacetimes/gravity, EFG

Spacetime as a nonholonomic manifold/ bundle $\mathbf{V} := (V, \mathcal{D})$ (Vrănceanu, 1926), or TM, with a non–integrable distribution $\mathcal{D}$.

Geometric data: Finsler (F : $\mathbf{N}, \mathbf{D}, \mathbf{g}$) and Riemannian $(\nabla, \mathbf{g})$

N–anholonomic frames: $\mathbf{e}_\nu = (\mathbf{e}_i = \partial_i - N_i^a \partial_a, e_a = \partial_a)$

Sasaki d–metric: ${}^F\mathbf{g} = {}^F g_{ij}(u) dx^i \otimes dx^j + {}^F g_{ab}(u)\ {}^c\mathbf{e}^a \otimes {}^c\mathbf{e}^b$,
for ${}^c\mathbf{e}^a = dy^a + {}^cN_i^a(u)\, dx^i$.
For $\tilde{\mathbf{D}}$, standard Riemannian, Ricci, Einstein d–tensors; h-/v–splitting.

N–adapted coef.: ${}^{Cart}\mathbf{D} = \tilde{\mathbf{D}} = (h\tilde{D}, v\tilde{D}) = \{\tilde{\Gamma}^\alpha_{\gamma\tau} = (\tilde{L}^i_{jk},\ \tilde{C}^a_{bc})\}$,

$$\tilde{L}^i_{jk} = \frac{1}{2}\ {}^F g^{ir}(\mathbf{e}_k\ {}^F g_{jr} + \mathbf{e}_j\ {}^F g_{kr} - \mathbf{e}_r\ {}^F g_{jk}),$$
$$\tilde{C}^a_{bc} = \frac{1}{2}\ {}^F g^{ad}(e_c\ {}^F g_{bd} + e_c\ {}^F g_{cd} - e_d\ {}^F g_{bc}).$$

Theorem: Equivalent (pseudo) Finsler & Riemannian theories if ${}^g\mathbf{D} = {}^g\nabla + {}^g\mathbf{Z}$, distortion determined by $\mathbf{g} = {}^F\mathbf{g}$.

Analogous Gravity and Lagrange–Finsler Geometry

Unified formalism for Riemann–Cartan, Finsler spaces and geometric mechanics.

Alternative works on analogous gravity. "Pseudo" (relativistic) geometric mechanics. $(-+++)$, local pseudo–Euclidian with $x^1 = i\, ^\circ x^1, i^2 = -1$.

Lagrange spaces: "Mechanical" modelling of gravitational interactions on semi–Riemannian manifolds **V**, or **E** $=$ **TM**, fundamental/generating Lagrange function $L(x,y)$:

$$^L g_{ab} = \frac{1}{2}\frac{\partial^2 L}{\partial y^a \partial y^b},\ det|g_{ab}| \neq 0.$$

Canonical N–connection

$$^L N^i_j(x,y) = \frac{\partial\ ^L G^i}{\partial y^j},\ \ ^L G^i = \frac{1}{4}\ ^L g^{ij}\left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i}\right)$$

nonlinear geodesic equations for $x^i(\tau)$, $y^i = \frac{dx^i}{d\ldots}$

Analogous Gravity and Lagrange–Finsler Geometry

Finsler/Lagrange modelling

Theorem: *Any Lagrange (Finsler) geometry can be modelled equivalently as a N–anholonomic Riemann manifold* $\mathbf{V}$, *and inversely, with canonically induced by* L (F) *d–metric structure*

$$
\begin{aligned}
{}^{L}\mathbf{g} &= {}^{L}g_{ij}(u)\, e^{i} \otimes e^{j} + {}^{L}g_{ab}(u)\, {}^{L}\mathbf{e}^{a} \otimes {}^{L}\mathbf{e}^{b} \\
e^{i} &= dx^{i},\ {}^{L}\mathbf{e}^{b} = dy^{b} + {}^{L}N_{j}^{b}(u) dx^{j};
\end{aligned}
$$

(not) N–adapted connections, ${}^{L}\widehat{\mathbf{D}}$; *equivalently,* ${}^{L}\nabla$.

Analogous Gravity and Lagrange–Finsler Geometry
Almost Kähler variables/models

in Lagrange–Finsler geometry, classical and quantum gravity, nonholonomic Ricci flows

Almost complex structure determined by the canonical N–connection: $\mathbf{J}(\mathbf{e}_i) = -e_i$ and $\mathbf{J}(e_i) = \mathbf{e}_i$

$L(x,y)$ induces a canonical 1–form ${}^{L}\omega = \frac{1}{2}\frac{\partial L}{\partial y^i} e^i$
${}^{L}\mathbf{g} \rightarrow$ canonical 2–f. ${}^{L}\theta(\mathbf{X},\mathbf{Y}) \doteqdot {}^{L}\mathbf{g}(\mathbf{J}\mathbf{X},\mathbf{Y}) = {}^{L}\mathbf{g}_{ij}(x,y)\mathbf{e}^i \wedge e^i$

Almost Kähler models of Lagrange–Finsler/Einstein spaces with ${}^{\theta}\widehat{\mathbf{D}} = \tilde{\mathbf{D}}$

$$ {}^{\theta}\widehat{\mathbf{D}}_{\mathbf{X}}\, {}^{L}\mathbf{g} = 0 \text{ and } {}^{\theta}\widehat{\mathbf{D}}_{\mathbf{X}}\, \mathbf{J} = 0. $$

Important for deformation quantization (Fedosov) of Einstein and Lagrange–Finsler/Hamilton–Cartan gravity.

Analogous Gravity and Lagrange–Finsler Geometry

Remarks:

1. $\exists$ a unique geometric formalism of nonholonomic deformations and analogous modeling of gravitational, Einstein and Finsler and "pseudo" mechanical models.
2. Key questions: for what types of connections we postulate the field equations and what class of nonholonomic constraints is involved?
3. Different Finsler d–connections (for instance) Chern's one $^{Ch}\Gamma^{\gamma}_{\ \alpha\beta} = \left(\widehat{L}^{i}_{jk}, \widehat{C}^{a}_{bc} = 0\right), \ ^{Ch}\mathbf{D}\ ^{F}\mathbf{g} \neq 0, \text{ but } \ ^{Ch}\mathbf{T} = 0.$
4. Nonmetricity is not compatible with standard physics: a. Definition of spinors; b. Conservation laws; c. Supersymmetric / noncommutative generalizations of Finsler like spaces; d. Exact solutions?

Principles and axioms of EFG

Principles: Similarly to GR with ${}^{g}\nabla$ on V construct **EFG:** with $\mathbf{g} \sim {}^{F}\mathbf{g}, \mathbf{N} \sim {}^{F}\mathbf{N}$ and ${}^{Cart}\mathbf{D}$ on TV, or $\mathbf{V}$.

1. Generalized equivalence principle: Ideas on Free Fall and Universality of Gravitational Redshift for ${}^{Cart}\mathbf{D}$.
2. Generalized Mach principle: quantum energy/motion encoded via $(\mathbf{N}, \mathbf{g}, \mathbf{D})$ for spacetime ether with y^a.
3. Principle of general covariance extended on $\mathbf{V}$, or TV, with "mixing of Finsler parametrizations".
4. Motion eqs and conservation laws: Nonholonomc Bianchi identities for ${}^{F}\mathbf{D}$; $\nabla_i T^{ij} = 0 \rightarrow \mathbf{D}_\alpha \Upsilon^{\alpha\beta} \neq 0$.
5. Einstein–Finsler gravitational field eqs for ${}^{F}\mathbf{D}$.
6. **Axiomatics:** Constructive–axiomatic appr. (Ehlers-Pirani –Schild, EPS axioms), paradigm "Lorentzian 4–manifold" in GR; nonholon. tangent bundle on "L ..." for EFG.

Gravitational field eqs in EFG

$\forall$ **D**, Einstein eqs: $\mathbf{E}_{\alpha\beta} = \Upsilon_{\alpha\beta}$,
h–/v–components, for $R_{ai} = R^b_{\ aib}$ and $R_{ia} = R^k_{\ ikb}$:

$$R_{ij} - \frac{1}{2}(R+S)g_{ij} = \Upsilon_{ij},$$
$$R_{ab} - \frac{1}{2}(R+S)h_{ab} = \Upsilon_{ab},$$
$$R_{ai} = \Upsilon_{ai}, \ R_{ia} = -\Upsilon_{ia},$$

Remark: For ${}^{Cart}\mathbf{D}$, general off–diagonal solutions for EFG, restrictions to GR, $\mathbf{g} = \underline{g}_{\alpha\beta}(u)\, du^\alpha \otimes du^\beta$,

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N^a_i N^b_j h_{ab} & N^e_j h_{ae} \\ N^e_i \ h_{be} & h_{ab} \end{bmatrix}, \text{where } \ N^a_i \neq A^a_{bi}(x) y^b$$

Claim: Compactification/trapping/warping mechanism on velocity/momenta for a "new" QG and LV phenomenology.

Gravitational field eqs in EFG

Levi–Civita and canonical d–connection

Levi–Civita connection $\nabla = \{ {}^{\mathbf{g}}\Gamma^{\gamma}_{\ \alpha\beta}\}, \quad T^{\alpha}_{\beta\gamma} = 0$ and $\nabla \mathbf{g} = 0$

Canonical d–connection $\widehat{\mathbf{D}} = \{ {}^{\mathbf{g}}\widehat{\Gamma}^{\gamma}_{\ \alpha\beta}\}$

$\widehat{\mathbf{D}}\mathbf{g} = 0$ and $h\widehat{\mathbf{T}}(hX, hY) = 0,\ v\widehat{\mathbf{T}}(vX, vY) = 0,\ {}^{\mathbf{g}}\Gamma^{\gamma}_{\ \alpha\beta} = {}^{\mathbf{g}}\widehat{\Gamma}^{\gamma}_{\ \alpha\beta} + {}^{\mathbf{g}}Z^{\gamma}_{\ \alpha\beta}$

Distortion ${}^{\mathbf{g}}Z^{\gamma}_{\ \alpha\beta}$ defined by $\mathbf{g}$, $\widehat{\Gamma}^{\gamma}_{\ \alpha\beta} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$,

$$\begin{aligned}
\widehat{L}^{i}_{jk} &= \frac{1}{2} g^{ir}\left(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}\right), \\
\widehat{L}^{a}_{bk} &= e_b(N^a_k) + \frac{1}{2} h^{ac}\left(\mathbf{e}_k h_{bc} - h_{dc}\, e_b N^d_k - h_{db}\, e_c N^d_k\right), \\
\widehat{C}^{i}_{jc} &= \frac{1}{2} g^{ik} e_c g_{jk},\ \widehat{C}^{a}_{bc} = \frac{1}{2} h^{ad}\left(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}\right).
\end{aligned}$$

Nontrivial d–torsion $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$: $\widehat{T}^{i}_{\ ja} = \widehat{C}^{i}_{jb},\ \widehat{T}^{a}_{ji} = -\Omega^{a}_{\ ji},\ \widehat{T}^{c}_{aj} = \widehat{L}^{c}_{aj} - e_a(N^c_j)$

If $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta} = 0,\ {}^{\mathbf{g}}\Gamma^{\gamma}_{\ \alpha\beta} = {}^{\mathbf{g}}\widehat{\Gamma}^{\gamma}_{\ \alpha\beta}$ even $\nabla \neq \widehat{\mathbf{D}}$

General Solutions in Gravity

Einstein eqs for the canonical d–connection

The Einstein equations for a d–metric $\mathbf{g}_{\beta\delta}$, also in GR, can be rewritten equivalently using $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ {}^{s}R = \Upsilon_{\beta\delta},$$

$$\widehat{L}_{aj}^{c} = e_a(N_j^c),\ \widehat{C}_{jb}^{i} = 0,\ \Omega_{\ ji}^{a} = 0,$$

$\widehat{\mathbf{R}}_{\ \beta\delta}$ for $\widehat{\Gamma}_{\ \alpha\beta}^{\gamma}$, ${}^{s}R = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\ \beta\delta}$ and $\Upsilon_{\beta\delta} \to \varkappa T_{\beta\delta}$ for $\widehat{\mathbf{D}} \to \nabla$.
(2+2) splitting, $(u^{\alpha} = (x^k, t, y^4)$, ansatz with Killing $\partial/\partial y^4$,

$$\begin{aligned} {}^{K}\mathbf{g} &= g_1(x^k)dx^1 \otimes dx^1 + g_2(x^k)dx^2 \otimes dx^2 \\ &+ h_3(x^k,t)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k,t)\mathbf{e}^4 \otimes \mathbf{e}^4 \end{aligned}$$

for $N_i^3 = w_i(x^k,t),\ N_i^4 = n_i(x^k,t)$,
$\mathbf{e}^3 = dt + w_i(x^k,t)dx^i,\ \mathbf{e}^4 = dy^4 + n_i(x^k,t)dx^i$

General Solutions in Gravity

Theorem 1 (Separation of Eqs)

The Einstein eqs for ansatz ${}^{K}\mathbf{g}$ and $\widehat{\mathbf{D}}$ are:

$$-\widehat{R}_1^1 = -\widehat{R}_2^2 = \frac{1}{2g_1 g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = \Upsilon_4(x^k)$$

$$-\widehat{R}_3^3 = -\widehat{R}_4^4 = \frac{1}{2h_3 h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}\right] = \Upsilon_2(x^k, t),$$

$$\widehat{R}_{3k} = \frac{w_k}{2h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}\right] + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_4^*}{2h_4} = 0,$$

$$\widehat{R}_{4k} = \frac{h_4}{2h_3} n_k^{**} + \left(\frac{h_4}{h_3} h_3^* - \frac{3}{2} h_4^*\right) \frac{n_k^*}{2h_3} = 0,$$

where $a^{\bullet} = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial t$.

Integration of (non)holonomic Einstein eq

Theorem 2 (Integral Varieties)

$$\begin{aligned} \ddot{\psi} + \psi'' &= 2\Upsilon_4(x^k) \\ h_4^* &= 2h_3h_4\Upsilon_2(x^i,t)/\phi^* \\ \beta w_i + \alpha_i &= 0 \\ n_i^{**} + \gamma n_i^* &= 0 \end{aligned}$$

$$\alpha_i = h_4^* \partial_i \phi, \beta = h_4^* \ \phi^*, \phi = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|, \gamma = \left(\ln \frac{|h_4|^{3/2}}{|h_3|} \right)^*, \ h_{3,4}^* \neq 0, \Upsilon_{2,4} \neq 0,$$

General solution: $g_1 = g_2 = e^{\psi(x^k)}, \ h_4 = {}^0h_4(x^k) \pm \ 2 \int \frac{(\exp[2 \ \phi(x^k, t)])^*}{\Upsilon_2} dt,$

$$h_3 = \pm \frac{1}{4} \left[\sqrt{|h_4^*(x^i, t)|} \right]^2 \exp[-2 \ \phi(x^k, t)]$$

$$w_i = -\partial_i \phi / \phi^*, \ n_k = {}^1n_k \left(x^i \right) + {}^2n_k \left(x^i \right) \int [h_3 / (\sqrt{|h_4|})^3] dt$$

LC conditions: $w_i^* = \mathbf{e}_i \ln |h_4|, \mathbf{e}_k w_i = \mathbf{e}_i w_k, \ n_i^* = 0, \ \partial_i n_k = \partial_k n_i$

Integration of (non)holonomic Einstein eq

General Solutions

Dependence on 4th coordinate via $\omega^2(x^j,t,y)$

$$\mathbf{g} = g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j,t,y)h_a(x^k,t)\mathbf{e}^a \otimes \mathbf{e}^a,$$
$$\mathbf{e}^3 = dy^3 + w_i(x^k,t)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k,t)dx^i,$$
$$\mathbf{e}_k\omega = \partial_k\omega + w_k\omega^* + n_k\partial\omega/\partial y = 0,$$

$\omega^2 = 1$ results in solutions with Killing symmetry.

N–deformations and exact solutions

'Polarizations' η_α and η_i^a, nonholonomic deformations,
$^\circ\mathbf{g} = [\ ^\circ g_i,\ ^\circ h_a,\ ^\circ N_k^a] \rightarrow\ ^\eta\mathbf{g} = [\ g_i, h_a, N_k^a]$.
Deformations of fundamental geometric structures:

$$^\eta\mathbf{g} = \eta_i(x^k,t)\ ^\circ g_i(x^k,t)dx^i \otimes dx^i + \eta_a(x^k,t)\ ^\circ h_a(x^k,t)\mathbf{e}^a \otimes \mathbf{e}^a,$$
$$\mathbf{e}^3 = dt + \eta_i^3(x^k,t)\ ^\circ w_i(x^k,t)dx^i,\ \mathbf{e}^4 = dy^4 + \eta_i^4(x^k,t)\ ^\circ n_i(x^k,t)dx^i.$$

Integration of (non)holonomic Einstein eq

Remarks

- "Almost" any solution of Einstein eqs, $g_{\alpha'\beta'}$, via $e_\alpha = e^{\alpha'}_{\ \alpha}(x^i, y^a)e_{\alpha'}$, $\mathbf{g}_{\alpha\beta} = e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta} g_{\alpha'\beta'}$, expressed $\mathbf{g}_{\alpha\beta} =$

$$\left| \begin{array}{cccc} g_1 + \omega^2({w_1}^2 h_3 + \omega^2({n_1}^2 h_4) & \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & \omega^2\ w_1 h_3 & \omega^2\ n_1 h_4 \\ \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & g_2 + \omega^2({w_2}^2 h_3 + {n_2}^2 h_4) & \omega^2\ w_2 h_3 & \omega^2\ n_2 h_4 \\ \omega^2\ w_1 h_3 & \omega^2\ w_2 h_3 & h_3 & 0 \\ \omega^2\ n_1 h_4 & \omega^2\ n_2 h_4 & 0 & h_4 \end{array} \right|$$

- Concept of general solutions for systems of nonlinear partial differential eqs? Topology, symmetries etc. Arbitrariness, uniqueness, sources?
- Complex/supersymmetric/ nonholonomic / quantum distributions – applications to modern gravity and physics
- Higher dimensions - "shell by shell". Almost Kähler structures etc, generalized (algebroid etc) symmetries. Nontrivial topology etc
- Exact solutions in astrophysics, cosmology: black ellipsoids/toruses, wormholes, solitons, Dirac waves, pp–waves etc

Nonholonomic Ricci Flows

Constrained Ricci Evolution

(Non) commutative/ supersymmetric Lagrange–Finsler, almost Kähler and nonholonomic Ricci flows

1. Families regular Lagrangians $L(u,\chi) = L(x,y,\chi)$ on TM, or $\mathbf{V}$
2. for instance, $\mathbf{g}_{\alpha\beta}$ as solutions of Einstein eqs $\mathbf{R}_{\alpha\beta} = \lambda\ \mathbf{g}_{\alpha\beta}$
3. $\mathbf{g}_{\alpha\beta}(\chi)$ as solutions of the Ricci flow eqs $\frac{\partial \mathbf{g}_{\alpha\beta}}{\partial \chi} = -2\mathbf{R}_{\alpha\beta}$ real parameter χ, Ricci tensor $\mathbf{R}_{\alpha\beta}$ for ∇ or any metric compatible connection $\mathbf{D}$, $\mathbf{Dg} = 0$, but torsion ${}^{g,D}\mathbf{T} \neq 0$
4.

$$\begin{aligned}
\text{N–adapted evolution:}\frac{\partial}{\partial \chi} g_{ii} &= -2\left[\widehat{R}_{ii} - \lambda g_{ii}\right] - h_{cc}\frac{\partial}{\partial \chi}(N_i^c)^2,\\
\frac{\partial}{\partial \chi} h_{aa} &= -2\left(\widehat{R}_{aa} - \lambda h_{aa}\right),\\
\widehat{R}_{\alpha\beta} &= 0,\ \text{for}\ \ \alpha \neq \beta
\end{aligned}$$

Ricci–Lagrange/–Finsler Evolution

(Semi)sprays and N–connections:

$$\frac{dy^a}{d\varsigma} + 2G^a(x,y) = 0,$$

curve $x^i(\varsigma),\ 0 \leq \varsigma \leq \varsigma_0$, when $y^i = dx^i/d\varsigma$.
Regular Lagrangian: $L(x,y) = L(x^i, y^a),\ {}^L g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$

$$N^a_i = \frac{\partial G^a}{\partial y^i},\ 4G^j = {}^L g^{ij}\left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i}\right),$$

$$\begin{aligned} {}^L\mathbf{g} &= {}^L g_{ij}(x,y)\left[e^i \otimes e^j + \mathbf{e}^i \otimes \mathbf{e}^j\right] \\ & \mathbf{e}^\alpha = [e^i = dx^i, \mathbf{e}^a = dy^a + N^a_i(x,y)dx^i]. \end{aligned}$$

Ricci–Lagrange/–Finsler Evolution

Hamilton's evolution eqs:

$$\frac{\partial g_{\alpha\beta}(\chi)}{\partial \chi} = -2\ {}_{\shortmid}R_{\alpha\beta}(\chi)$$

for a set of (semi) Riemannian metrics $g_{\alpha\beta}(\chi)$, real parameter χ, Ricci tensors ${}_{\shortmid}R_{\alpha\beta}(\chi)$ for the Levi–Civita connection.

Perelman's functionals for flows of Riemannian metrics

$$
\begin{aligned}
{}_{\shortmid}\mathcal{F}(L,f) &= \int_{\mathbf{V}} \left({}_{\shortmid}R + |\nabla f|^2 \right) e^{-f}\, dV, \\
{}_{\shortmid}\mathcal{W}(L,f,\tau) &= \int_{\mathbf{V}} \left[\tau \left({}_{\shortmid}R + |\nabla f| \right)^2 + f - 2n \right] \mu\, dV,
\end{aligned}
$$

volume form of ${}^{L}\mathbf{g}$, dV, integration over compact $\mathbf{V}$, function f for gradient flows with different measures, scalar curvature for ∇, ${}_{\shortmid}R$. For $\tau > 0$, $\int_{\mathbf{V}} \mu dV = 1$, $\mu = (4\pi\tau)^{-n} e^{-f}$.

Claim: For Lagrange spaces, Perelman's functionals for $\widehat{\mathbf{D}}$, $\widehat{\mathcal{F}}(L,\widehat{f}), \widehat{\mathcal{W}}(L,\widehat{f},\tau)$ are

$$\widehat{\mathcal{F}} = \int_{\mathbf{V}} \left(R + S + \left|\widehat{\mathbf{D}}\widehat{f}\right|^2 \right) e^{-\widehat{f}}\, dV,$$

$$\widehat{\mathcal{W}} = \int_{\mathbf{V}} \left[\widehat{\tau} \left(R + S + \left|{}^{h}D\widehat{f}\right| + \left|{}^{v}D\widehat{f}\right| \right)^2 + \widehat{f} - 2n \right] \widehat{\mu}\, dV,$$

R and S are h- and v–components of curvature scalar of $\widehat{\mathbf{D}} = ({}^{h}D, {}^{v}D)$, $\left|\widehat{\mathbf{D}}\widehat{f}\right|^2 = \left|{}^{h}D\widehat{f}\right|^2 + \left|{}^{v}D\widehat{f}\right|^2$, $\widehat{f}$ satisfies $\int_{\mathbf{V}} \widehat{\mu} dV = 1$ for $\widehat{\mu} = (4\pi\tau)^{-n} e^{-\widehat{f}}$ and $\tau > 0$.

Proofs for N–adapted evolution eqs

Theorem: If a Lagrange (Finsler) metric ${}^{L}\mathbf{g}(\chi)$ and functions $\widehat{f}(\chi)$ and $\widehat{\tau}(\chi)$ evolve for $\frac{\partial \widehat{\tau}}{\partial \chi} = -1$ and constant $\int_{\mathbf{V}} (4\pi\widehat{\tau})^{-n} e^{-\widehat{f}} dV$ as solutions of

$$\begin{aligned} \frac{\partial \underline{g}_{ij}}{\partial \chi} &= -2\underline{\widehat{R}}_{ij}, \ \frac{\partial \underline{g}_{ab}}{\partial \chi} = -2\underline{\widehat{R}}_{ab}, \\ \frac{\partial \widehat{f}}{\partial \chi} &= -\widehat{\Delta}\widehat{f} + \left|\widehat{\mathbf{D}}\widehat{f}\right|^2 - R - S + \frac{n}{\widehat{\tau}}, \end{aligned}$$

then $\frac{\partial}{\partial \chi}\widehat{\mathcal{W}}({}^{L}\mathbf{g}(\chi), \widehat{f}(\chi), \widehat{\tau}(\chi)) = 2\int_{\mathbf{V}} \widehat{\tau}[|\widehat{R}_{ij} + D_i D_j \widehat{f} - \frac{1}{2\widehat{\tau}} g_{ij}|^2$

$+|\widehat{R}_{ab} + D_a D_b \widehat{f} - \frac{1}{2\widehat{\tau}} g_{ab}|^2](4\pi\widehat{\tau})^{-n} e^{-\widehat{f}} dV.$

Corollary: The evolution, for all $\tau \in [0, \tau_0)$, of N–adapted frames $\mathbf{e}_\alpha(\tau) = \mathbf{e}_\alpha^{\ \underline{\alpha}}(\tau, u)\partial_{\underline{\alpha}}$ is defined by

$$\mathbf{e}_\alpha^{\ \underline{\alpha}}(\tau, u) = \left[\begin{array}{cc} e_i^{\ \underline{i}}(\tau, u) & N_i^b(\tau, u)\ e_b^{\ \underline{a}}(\tau, u) \\ 0 & e_a^{\ \underline{a}}(\tau, u) \end{array} \right],$$

with ${}^L g_{ij}(\tau) = e_i^{\ \underline{i}}(\tau, u)\ e_j^{\ \underline{j}}(\tau, u)\eta_{\underline{ij}}$ subjected to eqs

$$\frac{\partial}{\partial \tau}\ e_\alpha^{\ \underline{\alpha}} = {}^L g^{\underline{\alpha\beta}}\ {}_{\shortmid}R_{\underline{\beta\gamma}}\ \ e_\alpha^{\ \underline{\gamma}}, \quad \text{for the Levi-Civita connection;}$$
$$\frac{\partial}{\partial \tau}\ e_\alpha^{\ \underline{\alpha}} = {}^L g^{\underline{\alpha\beta}}\ \widehat{R}_{\underline{\beta\gamma}}\ \ e_\alpha^{\ \underline{\gamma}}, \quad \text{for the canonical d–connection.}$$

Finsler–branes & cosmological solutions

Nonholon. trapping solutions (cosmology, with $h_3(x^i, y^3 = t))$:

$$\mathbf{g} = g_1 dx^1 \otimes dx^1 + g_2 dx^2 \otimes dx^2 + h_3 \mathbf{e}^3 \otimes \mathbf{e}^3 + h_4 \mathbf{e}^4 \otimes \mathbf{e}^4 + (l_P)^2 \frac{\overline{h}}{\phi^2} [\ {}^q h_5 \mathbf{e}^5 \otimes \mathbf{e}^5 + {}^q h_6 \mathbf{e}^6 \otimes \mathbf{e}^6 + {}^q h_7 \mathbf{e}^7 \otimes \mathbf{e}^7 + {}^q h_8 \mathbf{e}^8 \otimes \mathbf{e}^8]$$

$$\mathbf{e}^3 = dy^3 + w_i dx^i, \mathbf{e}^4 = dy^4 + n_i dx^i,\ \mathbf{e}^5 = dy^5 + {}^1 w_i dx^i,$$
$$\mathbf{e}^6 = dy^6 + {}^1 n_i dx^i,\ \mathbf{e}^7 = dy^7 + {}^2 w_i dx^i,\ \mathbf{e}^8 = dy^8 + {}^2 n_i dx^i.$$

$$\phi^2(y^5) = \frac{3\epsilon^2 + a(y^5)^2}{3\epsilon^2 + (y^5)^2} \text{ and } l_P \sqrt{|\overline{h}(y^5)|} = \frac{9\epsilon^4}{\left[3\epsilon^2 + (y^5)^2\right]^2},$$

N–connection coefficients determined by sources

$$\begin{aligned} {}^h\Lambda(x^i) &= \widetilde{\Upsilon}_4 + \widetilde{\Upsilon}_6 + \widetilde{\Upsilon}_8,\ {}^v\Lambda(x^i, v) = \widetilde{\Upsilon}_2 + \widetilde{\Upsilon}_6 + \widetilde{\Upsilon}_8, \\ {}^5\Lambda(x^i, y^5) &= \widetilde{\Upsilon}_2 + \widetilde{\Upsilon}_4 + \widetilde{\Upsilon}_8,\ {}^7\Lambda(x^i, y^5, y^7) = \widetilde{\Upsilon}_2 + \widetilde{\Upsilon}_4 + \widetilde{\Upsilon}_6. \end{aligned}$$

Conclusions

- Almost all models of QG with nonlinear dispersions can be geometrized as certain Finsler spacetimes.
- Natural/ Canonical Principles for metric compatible EFG generalizing the GR on $TV, \nabla \rightarrow {}^{Cart}\mathbf{D}$.
- Finsler branes, trapping: "new" QG/ LV phenomenology.
- **Outlook** (recently developed, under elaboration):
  - EFG is almost completely integrable, can be quantized as almost Kähler–Fedosov/ A–brane geometries, and renormalizable for bi–connection/gauge gravity models.
  - Finsler for black holes (ellipsoids, toruses, holes, wormholes, solitons); anisotropic cosmological models (off–diagonal inflation, dark energy/matter etc).
  - Noncommutative/ Ricci–Finsler flows, emergent (non) commutative Lagrange–Finsler analogous gravity and quantization, Clifford–Finsler algebroids etc.