Homogeneous geodesics in homogeneous affine manifolds

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Definition

Let (M, ∇) be a homogeneous affine manifold.
A geodesic is homogeneous if it is an orbit of an one-parameter group of affine diffeomorphisms. (Here the canonical parameter of the group need not be the affine parameter of the geodesic.)
An affine g.o. manifold is a homogeneous affine manifold (M, ∇) such that each geodesic is homogeneous.

Lemma

Let $M = G/H$ be a homogeneous space with a left-invariant affine connection ∇. Then each regular curve which is an orbit of a 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M.

Definition

Let (M, ∇) be a manifold with an affine connection.
A vector field X on M is called a Killing vector field if

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0$$

is satisfied for arbitrary vector fields Y, Z.

Lemma

Let (M, ∇) be a homogeneous affine manifold and $p \in M$. There exist $n = \dim(M)$ affine Killing vector fields which are linearly independent at each point of some neighbourhood $\mathcal{U}$ of p.

Definition

A nonvanishing smooth vector field Z on M is geodesic along its regular integral curve γ if $\gamma(t)$ is geodesic up to a possible reparametrization. If all regular integral curves of Z are geodesics up to a reparametrization, then the vector field Z is called a geodesic vector field.

For example, a round two-sphere with the corresponding Levi-Civita connection does *not* admit any geodesic affine Killing vector field. Still, all geodesics are homogeneous.

Lemma

Let Z be a nonvanishing Killing vector field on $M = (G/H, \nabla)$.
1) *Z is geodesic along its integral curve γ if and only if*

$$\nabla_{Z_{\gamma(t)}} Z = k_\gamma \cdot Z_{\gamma(t)}$$

holds along γ. Here $k_\gamma \in \mathbb{R}$ is a constant.
2) *Z is a geodesic vector field if and only if*

$$\nabla_Z Z = k \cdot Z$$

holds on M. Here k is a smooth function on M which is constant along integral curves of Z.

$\dim(M) = 2$

Theorem (Opozda; Arias-Marco, Kowalski)

Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold $\mathcal{M}$. Then, either ∇ is locally a Levi-Civita connection of the unit sphere or, in a neighbourhood $\mathcal{U}$ of each point $m \in \mathcal{M}$, there is a system (u, v) of local coordinates and constants A, B, C, D, E, F, G, H such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:

$$\begin{array}{lll}
\underline{\textit{TypeA}}: & \nabla_{\partial_u}\partial_u = A\,\partial_u + B\,\partial_v, & \nabla_{\partial_u}\partial_v = C\,\partial_u + D\,\partial_v, \\
 & \nabla_{\partial_v}\partial_u = E\,\partial_u + F\,\partial_v, & \nabla_{\partial_v}\partial_v = G\,\partial_u + H\,\partial_v, \\
\underline{\textit{TypeB}}: & \nabla_{\partial_u}\partial_u = \frac{A}{u}\,\partial_u + \frac{B}{u}\,\partial_v, & \nabla_{\partial_u}\partial_v = \frac{C}{u}\,\partial_u + \frac{D}{u}\,\partial_v, \\
 & \nabla_{\partial_v}\partial_u = \frac{E}{u}\,\partial_u + \frac{F}{u}\,\partial_v, & \nabla_{\partial_v}\partial_v = \frac{G}{u}\,\partial_u + \frac{H}{u}\,\partial_v.
\end{array}$$

Connections of type A

- Let us have a connection ∇ with constant Christoffel symbols. The operators ∂_u, ∂_v are affine Killing vector fields.
- A general vector field $X = x\,\partial_u + y\,\partial_v$ satisfies the condition $\nabla_X X = kX$ if it holds

$$\begin{aligned} Ax^2 + (C+E)xy + Gy^2 &= k\,x, \\ Bx^2 + (D+F)xy + Hy^2 &= k\,y. \end{aligned} \tag{1}$$

- By the elimination of the factor k we obtain

$$Bx^3 - (A-D-F)x^2y - (C+E-H)xy^2 - Gy^3 = 0.$$

- A sufficient condition for a vector field $X = x\,\partial_u + y\,\partial_v$ to be geodesic is that the pair (x, y) satisfies this condition.
- For any connection of type A, a geodesic Killing field (and at least one homogeneous geodesic) exist.

Affine g.o. manifold

Theorem

For $(\mathbb{R}^2, \nabla)$ *to be an affine g.o. manifold, it is sufficient that*

$$B = 0,\ A = D + F,\ G = 0,\ H = C + E.$$

- In this case, the equations (1) give

$$Ax + Hy = k,$$

<u>k is nonzero</u> in general and geodesics must be reparametrized.

Connections of type B

- The globally homogeneous manifold $\mathcal{U} = \{\mathbb{R}(u, v) \mid u > 0\}$. The general Killing vector field is $X = x\partial_v + y(u\partial_u + v\partial_v)$.
- The equality $\nabla_{X_{\gamma(t)}} X = k_\gamma \cdot X_{\gamma(t)}$ gives

$$\begin{aligned} ((A+1)c_1^2 + (C+E)c_1c_2 + Gc_2^2)y &= k_\gamma c_1^2, \\ (Bc_1^2 + (D+F+1)c_1c_2 + Hc_2^2)y &= k_\gamma c_1c_2. \end{aligned} \quad (2)$$

- By the elimination of k_γ we obtain

$$B\, c_1^3 - (A - D - F)\, c_1^2\, c_2 - (C + E - H)\, c_1\, c_2^2 - G\, c_2^3 = 0.$$

- $(\mathcal{U}, \nabla)$ admits at least one homogeneous geodesic through each point.
- Homogeneous geodesics are the integral curves of Killing vector fields which are not geodesic.
- In general, connections of type B do not admit any geodesic Killing vector fields.

Affine g.o. manifold

Theorem

If it holds

$$B = 0,\ A = D + F,\ G = 0,\ H = C + E,$$

then for any $(x, y) \neq (0, 0)$ *the Killing vector field*

$$X = x\partial_v + y(u\partial_u + v\partial_v)$$

is geodesic. $(\mathcal{U}, \nabla)$ *is an affine g.o. manifold and any homogeneous geodesic is the integral curve of a geodesic Killing vector field.*

- In this case, the equations (2) give us

$$((A+1)c_1 + Hc_2)y \quad = \quad k_\gamma c_1.$$

- For a given geodesic Killing field, different geodesics must be reparametrized by <u>different k_γ</u>.

Homogeneous geodesics in dimension 3

$(\mathbb{R}^3, \nabla)$
connection ∇ with constant Christoffel symbols
group $\mathbb{R}^3$ acting on it by the translations

$$\Gamma^i_{11} = A_i, \quad \Gamma^i_{22} = B_i, \quad \Gamma^i_{33} = C_i,$$
$$\Gamma^i_{12} = \Gamma^i_{21} = E_i, \quad \Gamma^i_{13} = \Gamma^i_{31} = F_i, \quad \Gamma^i_{23} = \Gamma^i_{32} = G_i.$$

The Killing vector field $X = x\,\partial_u + y\,\partial_v + z\,\partial_w$
satisfies the condition $\nabla_X X = kX$ if it holds

$$\begin{aligned} x^2 A_1 + y^2 B_1 + z^2 C_1 + 2\,xy E_1 + 2\,xz F_1 + 2\,yz G_1 &= k\,x, \\ x^2 A_2 + y^2 B_2 + z^2 C_2 + 2\,xy E_2 + 2\,xz F_2 + 2\,yz G_2 &= k\,y, \\ x^2 A_3 + y^2 B_3 + z^2 C_3 + 2\,xy E_3 + 2\,xz F_3 + 2\,yz G_3 &= k\,z. \end{aligned}$$

Families of homogeneous connections on H_3 or on $E(1,1)$
lead to similar equations.

Existence of homogeneous geodesics in dimension 3

Theorem

Let ∇ be a connection with constant Christoffel symbols on $\mathbb{R}^3$. $(\mathbb{R}^3, \nabla)$ admits a geodesic Killing vector field.

Proof. Recall that the Killing vector field $X = x\,\partial_u + y\,\partial_v + z\,\partial_w$ satisfies the condition $\nabla_X X = kX$ if it holds

$$\begin{aligned} x^2 A_1 + y^2 B_1 + z^2 C_1 + 2\,xy\,E_1 + 2\,xz F_1 + 2\,yz\,G_1 &= k\,x, \\ x^2 A_2 + y^2 B_2 + z^2 C_2 + 2\,xy\,E_2 + 2\,xz F_2 + 2\,yz\,G_2 &= k\,y, \\ x^2 A_3 + y^2 B_3 + z^2 C_3 + 2\,xy\,E_3 + 2\,xz F_3 + 2\,yz\,G_3 &= k\,z. \end{aligned}$$

- Sphere S^2 in T_pM, vectors $X = (x, y, z)$ with the norm 1.
- Denote $\underline{v(X) = \nabla_X X}$ and $t(X) = v(X) - \langle v(X), X\rangle X$, then $t(X) \perp X$ and $X \mapsto t(X)$ defines a vector field on S^2.
- According to the Hair-Dressing Theorem for sphere, there is $\bar{X} \in T_pM$ such that $t(\bar{X}) = 0$.
- We see $v(\bar{X}) = k\bar{X}$, hence $\underline{\nabla_{\bar{X}} \bar{X} = k\bar{X}}$.

Existence of homogeneous geodesics in odd dimensions

Theorem

Let $M = (G/H, \nabla)$ be a homogeneous affine manifold of odd dimension n and $p \in M$. There exists a homogeneous geodesic through p.

Proof. Killing vector fields $K_1, \dots, K_n$ independent near p,
$B = \underline{\{K_1(p), \dots, K_n(p)\} \text{ basis of } T_pM}$,
$X \in T_pM$, $X = (x_1, \dots x_n)$ in B,
$\underline{X^* = x_1K_1 + \cdots + x_nK_n}$ and an $\underline{\text{integral curve}}$ γ $\underline{\text{of } X^* \text{ through } p}$.
S^{n-1} in T_pM of vectors $X = (x_1, \dots, x_n)$ with the norm 1.
Denote $\underline{v(X) = \nabla_{X^*_{\gamma(t)}} X^*|_{t=0}}$ and $t(X) = v(X) - \langle v(X), X\rangle X$,
then $t(X) \perp X$ and $X \mapsto t(X)$ defines a vector field on S^{n-1}.
Again, there is $\bar{X} \in T_pM$ such that $t(\bar{X}) = 0$.
We obtain $v(\bar{X}) = k_\gamma \bar{X}$, where $k_\gamma = \langle v(\bar{X}), \bar{X}\rangle$ is a constant,
$\underline{\nabla_{\bar{X}^*_\gamma} \bar{X}^* = k_\gamma \bar{X}^*_\gamma}$ and γ is homogeneous geodesic. □

Preliminaries on differential topology

Let $f\colon M \to N$ be a smooth map between manifolds of the same dimension.
We say that $x \in M$ is a *regular point* of f if the derivative df_x is nonsingular. In this case, f maps a neighborhood of x diffeomorphically onto an open set in N.
The point $y \in N$ is called a *regular value* if $f^{-1}(y)$ contains only regular points.

- If M is compact and $y \in N$ is a regular value, then $f^{-1}(y)$ is a finite set (possibly empty).

For compact M, smooth map $f\colon M \to N$ and a regular value $y \in N$, we define $\#f^{-1}(y)$ to be the number of points in $f^{-1}(y)$.

- $\#f^{-1}(y)$ is locally constant as a function of y, where y ranges through regular values.

Points, or values, respectively, which are not regular are *critical*.

Theorem (Morse, Sard)

Let $f\colon U \to \mathbb{R}^n$ be a smooth map, defined on an open set $U \subset \mathbb{R}^m$ and let C be the set of critical points; that is the set of all $x \in U$ with $\mathrm{rank}(df_x) < n$. Then the image $f(C) \subset \mathbb{R}^n$ has measure zero.

Corollary (Brown)

The set of regular values of a smooth map $f\colon M \to N$ is everywhere dense in N.

Theorem

Let M and N be manifolds of the same dimension, M compact without boundary, N connected and $f\colon M \to N$ smooth mapping. If y and z are regular values of f, then

$$\#f^{-1}(y) = \#f^{-1}(z) \qquad (\mathrm{mod} 2).$$

This common residue class (called mod 2 degree of f) depends only on the smooth homotopy class of f.

Let M and N be oriented n-dimensional manifolds without boundary, M compact and N connected.
Let $f\colon M \to N$ be a smooth map and $x \in M$ a regular point of f, hence $df_x\colon T_xM \to T_{f(x)}N$ is a linear isomorphism.
Define the *sign of df_x* to be $+1$ or -1 according as df_x preserves or reverses orientation. For any regular value $y \in N$ define

$$\deg(f, y) \quad = \quad \sum_{x\in f^{-1}(y)} \operatorname{sign}(df_x).$$

- Again, $\deg(f, y)$ is a locally constant function of y and it is defined on a dense open subset of N.
- The integer $\deg(f, y)$ does not depend on the choice of regular value y and it is called *degree of f*.
- If f is homotopic to g, then $\deg(f) = \deg(g)$.

- Reflection $r_i \colon S^n \to S^n$ defined by

$$r_i(x_1, \ldots, x_n) \;=\; (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$$

  is an orientation reversing diffeomorphism with degree -1.
- The antipodal map $x \mapsto -x$ of S^n has degree $(-1)^{n+1}$, because it is the composition of $n+1$ reflections.
- Any map $f \colon S^n \to S^n$ without fixed points has degree $(-1)^{n+1}$, because it is homotopic to the antipodal map. The homotopy is for example

$$\varphi(x,t) \;=\; \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

  Clearly, $\varphi(x,0) = f(x)$ and $\varphi(x,1)$ is the antipodal map.

Existence of homogeneous geodesics in any dimension

We refine the proof of previous Theorem to arbitrary dimension.
Recall that $X \mapsto t(X)$ defines a smooth vector field on S^{n-1}.
Assume now that $t(X) \neq 0$ everywhere.
Putting $f(X) = t(X)/\|t(X)\|$, we obtain a smooth map $f\colon S^{n-1} \to S^{n-1}$ without fixed points.
According to a well-known statement from differential topology, the degree of f is $\deg(f) = (-1)^n$.
On the other hand, we have $v(X) = v(-X)$ and hence $f(X) = f(-X)$ for each X.
If Y is a regular value of f, then the inverse image $f^{-1}(Y)$ consists of even number of elements. Hence $\deg(f)$ is an even number, which is a contradiction.
This implies that there is $\bar{X} \in T_pM$ such that $t(\bar{X}) = 0$ and again, a homogeneous geodesic exists.

Existence of homogeneous geodesics

Theorem

Let $M = (G/H, \nabla)$ *be a homogeneous affine manifold and* $p \in M$.
Then M *admits a homogeneous geodesic through* p.

Theorem

Let $M = (G/H, g)$ *be a homogeneous pseudo-Riemannian manifold (not necessarily reductive) and* $p \in M$.
Then M *admits a homogeneous geodesic through* p.

Equiaffine connections

Definition

Let (M, ∇) be a torsion-free affine manifold. The affine connection ∇ is said to be *equiaffine* if there exists a nonvanishing *n*-form ω which is parallel with respect to ∇.

- A simply connected manifold (M, ∇) is equiaffine if and only if the Ricci tensor Ric^∇ is symmetric.
- Any homogeneous connection with constant Christoffel symbols is equiaffine.
- The group $\mathrm{SL}(2, \mathbb{R})$ acts naturally on the tangent space of each point and this action induces the natural action on the space of connections with constant Christoffel symbols.
- The only well-known polynomial invariant with respect to this action is the determinant of the Ricci matrix.

Representation of $SL(2,\mathbb{R})$ on $\mathbb{R}^6$

$\mathcal{H}$ ... set of torsion-free connections
with constant Christoffel symbols on $\mathbb{R}^2$

$$\begin{aligned}
\Gamma^1_{11} = A_1, \quad \Gamma^1_{12} = \Gamma^1_{21} = E_1, \quad \Gamma^1_{22} = B_1,\\
\Gamma^2_{11} = A_2, \quad \Gamma^1_{12} = \Gamma^2_{21} = E_2, \quad \Gamma^2_{22} = B_2.
\end{aligned}$$

$SL(2,\mathbb{R})$ is acting on frames in the plane $\mathbb{R}^2[u,v]$
and it induces the action on $\mathcal{H} = \mathbb{R}^6[A_1, A_2, B_1, B_2, E_1, E_2]$

$$\begin{aligned}
\bar{A}_1 &= ad^2A_1 + bd^2A_2 + ac^2B_1 + bc^2B_2 - 2\,acd\,E_1 - 2\,bcd\,E_2,\\
\bar{A}_2 &= cd^2A_1 + d^3A_2 + c^3B_1 + c^2d\,B_2 - 2\,c^2d\,E_1 - 2\,cd^2E_2,\\
\bar{B}_1 &= ab^2A_1 + b^3A_2 + a^3B_1 + a^2bB_2 - 2\,a^2bE_1 - 2\,ab^2E_2,\\
\bar{B}_2 &= b^2cA_1 + b^2dA_2 + a^2cB_1 + a^2dB_2 - 2\,abc\,E_1 - 2\,abd\,E_2,\\
\bar{E}_1 &= -abd\,A_1 - b^2dA_2 - a^2cB_1 - abc\,B_2\\
&\qquad +a(bc+ad)E_1 + b(bc+ad)E_2,\\
\bar{E}_2 &= -bcd\,A_1 - bd^2A_2 - ac^2B_1 - acd\,B_2\\
&\qquad +c(bc+ad)E_1 + d(bc+ad)E_2.
\end{aligned}$$

Invariants of the representation of $SL(2,\mathbb{R})$ on $\mathcal{H}$

- The determinant of the Ricci matrix is

$$\begin{aligned} I_1 &= (A_2E_1 + E_2^2 - A_1E_2 - A_2B_2)(B_1E_2 - B_2E_1 - A_1B_1 + E_1^2) \\ &\quad -(A_2B_1 - E_1E_2)^2. \end{aligned}$$

- Vector field $X = x\,\partial_u + y\,\partial_v$ satisfies $\nabla_X X = 0$ if

$$\begin{aligned} x^2A_1 + y^2B_1 + 2xyE_1 &= 0, \\ x^2A_2 + y^2B_2 + 2xyE_2 &= 0. \end{aligned}$$

  The resultant of these polynomials is

$$I_2 = 4\,(A_1\,E_2 - E_1\,A_2)\,(B_1\,E_2 - E_1\,B_2) + (A_1\,B_2 - A_2\,B_1)^2\,.$$

Invariants of the representation of $SL(2,\mathbb{R})$ on $\mathcal{H}$

- This representation admits 3 independent invariants. Using the computer, we found

$$\begin{aligned} I_3 &= (A_1^2 + A_1E_2 + A_2B_2 + A_2E_1)(A_1B_1 + B_1E_2 + B_2^2 + B_2E_1) \\ &\quad -(A_1E_1 + B_2E_2 + 2E_1E_2)^2. \end{aligned}$$

Theorem

Polynomials I_1, I_2, I_3 form a Hilbert basis of scalar invariants of the representation ρ of $SL(2,\mathbb{R})$.

Representation of $SL(2,\mathbb{R})$ on $\mathbb{R}^9$

$\mathcal{H}'$ ... space of torsion-free affine connections with constant Christoffel symbols on $\mathbb{R}^3[u,v,w]$

$$\begin{aligned} \Gamma^i_{11} &= A_i, & \Gamma^i_{22} &= B_i, & \Gamma^i_{33} &= C_i,\\ \Gamma^i_{12} = \Gamma^i_{21} &= E_i, & \Gamma^i_{13} = \Gamma^i_{31} &= F_i, & \Gamma^i_{23} = \Gamma^i_{32} &= G_i. \end{aligned}$$

Representation ρ' of $SL(2,\mathbb{R})$ on $\mathcal{H}' = \mathbb{R}^9[A_i, B_i, E_i]$

$$\begin{aligned}
\bar{A}_1 &= ad^2A_1 + bd^2A_2 + ac^2B_1 + bc^2B_2 - 2\,acd\,E_1 - 2\,bcd\,E_2,\\
\bar{A}_2 &= cd^2A_1 + d^3A_2 + c^3B_1 + c^2d\,B_2 - 2\,c^2d\,E_1 - 2\,cd^2E_2,\\
\bar{A}_3 &= d^2A_3 + c^2B_3 - 2\,cd\,E_3,\\
\bar{B}_1 &= ab^2A_1 + b^3A_2 + a^3B_1 + a^2bB_2 - 2\,a^2bE_1 - 2\,ab^2E_2,\\
\bar{B}_2 &= b^2cA_1 + b^2d\,A_2 + a^2cB_1 + a^2d\,B_2 - 2\,abc\,E_1 - 2\,abd\,E_2,\\
\bar{B}_3 &= b^2A_3 + a^2B_3 - 2\,abE_3,\\
\bar{E}_1 &= -abd\,A_1 - b^2d\,A_2 - a^2c\,B_1 - abc\,B_2\\
&\qquad + a(bc+ad)E_1 + b(bc+ad)E_2,\\
\bar{E}_2 &= -bcd\,A_1 - bd^2A_2 - ac^2B_1 - acd\,B_2\\
&\qquad + c(bc+ad)E_1 + d(bc+ad)E_2,\\
\bar{E}_3 &= -bd\,A_3 - ac\,B_3 + (ad+bc)\,E_3.
\end{aligned}$$

Invariants of the representation of $SL(2,\mathbb{R})$ on $\mathcal{H}'$

- The representation space $\mathcal{H}'$ of $\underline{\rho' \text{ decomposes}}$

$$\begin{aligned}\mathcal{H}' &= \mathbb{R}^9[A_i, B_i, E_i] = \\ &= \mathbb{R}^6[A_1, A_2, B_1, B_2, E_1, E_2] \oplus \mathbb{R}^3[A_3, B_3, E_3] = \\ &= \mathcal{H} \oplus \widetilde{\mathcal{H}}\end{aligned}$$

  and we can denote $\rho' = \rho \oplus \tilde{\rho}$.
- The invariants of ρ are I_1, I_2, I_3.
- The $\underline{\text{invariant of } \tilde{\rho}}$ is

$$I_4 = A_3 B_3 - E_3^2.$$

Invariants of the representation of $SL(2,\mathbb{R})$ on $\mathcal{H}'$

An invariant related with homogeneous geodesics is

$$
\begin{aligned}
I_5 \;=\;& {A_1}^2 A_3\, {B_3}^2 + {A_2}^2 {B_3}^3 + {A_3}^3 {B_1}^2 \\
&+ A_1 \Big(2\, A_2\, {B_3}^2 E_3 - 2\, {A_3}^2 B_1\, B_3 \\
&\qquad + A_3\, \left(4\, B_1\, {E_3}^2 + 2\, B_2\, B_3\, E_3 - 4\, {B_3}^2 E_2 - 4\, B_3\, E_1\, E_3\right)\Big) \\
&+ A_2 \Big(A_3\, \left(-6\, B_1\, B_3\, E_3 - 2\, B_2\, {B_3}^2 + 4\, {B_3}^2 E_1\right) \\
&\qquad + 8\, B_1\, {E_3}^3 + 4\, B_2\, B_3\, {E_3}^2 - 4\, {B_3}^2 E_2\, E_3 - 8\, B_3\, E_1\, {E_3}^2\Big) \\
&+ {A_3}^2 \Big(2 B_1\, \left(B_2\, E_3 + 2\, B_3\, E_2 - 2\, E_1\, E_3\right) \\
&\qquad + {B_2}^2 B_3 - 4\, B_2\, B_3\, E_1 + 4\, B_3\, {E_1}^2\Big) \\
&+ A_3 \Big(-8\, B_1\, E_2\, {E_3}^2 - 4\, B_2\, B_3\, E_2\, E_3 + 4\, {B_3}^2 {E_2}^2 + 8\, B_3\, E_1\, E_2\, E_3\Big).
\end{aligned}
$$

Invariants of the representation of $SL(2,\mathbb{R})$ on $\mathcal{H}'$

Open problem. The representation ρ' on $\mathbb{R}^9$ has 6 independent invariants. We know the invariants $I_1, \ldots, I_5$. Finding the last invariant of this representation and its geometrical meaning remains an open problem.

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