

Generalized Weierstrass representations for higher dimensional pseudo-Riemannian manifolds

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Introduction

Classical Weierstraß representations

Theorem (Weierstraß 1866)

Let M be a Riemannian surface. The two following statements are equivalent :

- 1) The map $F = (F_1, F_2, F_3) : M \rightarrow \mathbb{R}^3$ is a *minimal conformal immersion*.
- 2) There exists a triple $\omega = (\omega_1, \omega_2, \omega_3)$ of *holomorphic* $(1,0)$ -forms on M such that
 - (i) $\sum_i \omega_i^2 = 0$, $\sum_i \omega_i \bar{\omega}_i = 0$
 - (ii) The forms $\Re \omega_i$ are *exact*.

satisfying

$$F(q) = \Re \int_p^q (\omega_1, \omega_2, \omega_3) + \text{Constant}.$$

- The F_i are harmonic : $0 = \Delta F_i$
- $F = \Re(f)$, $f : M \rightarrow \mathbb{C}^3$ (holomorphic representative),
- $\omega_i = \phi_i dz$ with $\phi := \frac{\partial f}{\partial z} = (\phi_1, \phi_2, \phi_3)$ in

$$Q = \{(z_1^2, z_2^2, z_3^2) \in \mathbb{C}^3 | z_1^2 + z_2^2 + z_3^2 = 0\},$$

- There exists an associated family $\{F_\theta = \theta \in \mathbb{R}\}$, $F_0 = F$ given by

$$F_\theta = \Re(e^{i\theta f})$$

of "isometric deformations".

Other results.

- **Kenmotsu** : Representation for CMC-surfaces in \mathbb{R}^3 .
- **Konopelchenko, Kenmotsu, Bobenko** : Representations of minimal and CMC surfaces in \mathbb{R}^n ,
- **Abresch, Kusner-Schmitt '99, Taimanov...** : spinorial Weierstraß representations,
- **Dajzer-Gromoll '85-'95, Eschenburg-Tribuzy-Ferreira,...** : Kähler submanifolds of spaceforms

Definition

Let $f : M^{2m} \rightarrow \mathbb{R}^n$ be a Kählerian immersion, α its second fundamental form.

For $\theta \in \mathbb{R}$, define :

$$R_\theta : TM \rightarrow TM, \quad R_\theta = e^{J\theta} = \cos(\theta)I + \sin(\theta)J,$$

An **associated family** of f is a one-parameter family of Kählerian immersions $f_\theta : M^{2m} \rightarrow \mathbb{R}^n$, $f_0 = f$, with second fundamental form α_{f_θ} such that

$$\psi_\theta(\alpha_{f_\theta}(X, Y)) = \alpha_\theta(X, Y) := \alpha(R_\theta X, R_\theta Y),$$

for some bundle isomorphisms $\psi_\theta : f_\theta^* TN \rightarrow f_0^* TN$.

Theorem

Let $f : M^{2m} \rightarrow \mathbb{R}^n$ be a Kählerian immersion. Then there exists an associated family of f if and only if f has *parallel pluri-mean curvature*.

Remark : Pluri-minimal immersions have an associated family.

Results

Higher dimensional representations

Consider \mathbb{R}^n with standard metric :

$$dx_1^2 + \dots + dx_p^2 + dx_{p+1}^2 + \dots + dx_n^2.$$

Let M^{2m} be complex, $f : M^{2m} \rightarrow \mathbb{R}^n$ a smooth map.

$$f(q) = \int_p^q df + f(p) = \Re \int_p^q (\omega_1, \dots, \omega_n) + f(p),$$

where $\{\omega_i\}_{i=1, \dots, p+q}$ are $(1, 0)$ -forms, $\Re \omega_i$ exact.

$$\omega_i(z_1, \dots, z_m) = \sum_j^m \Omega_{ij}(z_1, \dots, z_m) dz_j, \quad \Omega_{ij}(z_1, \dots, z_m) = \left(\frac{\partial f_i}{\partial z_j} \right)_{ij}.$$

Define the maps

$$\begin{aligned} \phi_j : U &\mapsto \mathbb{C}^n \\ (z_1, \dots, z_m) &\rightarrow \frac{\partial f}{\partial z_j} = \frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j}, j = 1, \dots, m. \end{aligned}$$

Theorem (L.)

Let M^{2m} be a complex manifold. Then the following two conditions are equivalent :

1. The map $f : M^{2m} \rightarrow \mathbb{R}^n$ is a **pluri-conformal** immersion.
2. There exist maps $\phi_j : U \rightarrow \mathbb{C}^n$, $j = 1, \dots, m$, satisfying the conditions
 - a) The vector $\phi_1(p), \dots, \phi_m(p)$ are **linearly independent** at every $p \in U$.
 - b) $\phi_i \cdot \phi_j = 0$, $i, j = 1 \dots m$,
 - c) $\phi_i \cdot \phi_j > 0$, $i, j = 1 \dots m$,
where " \cdot " is the product $v \cdot w := \sum_{i=1}^n v_i w_i$.
 - d) $\frac{\partial \phi_j}{\partial z_k} = \frac{\partial \phi_k}{\partial z_j}$, $\frac{\partial \phi_j}{\partial \bar{z}_k} = \frac{\partial \overline{\phi_k}}{\partial \bar{z}_j}$ $i, j = 1 \dots n$.

Proposition

A pluri-conformal immersion is *pluriminimal* if and only if the maps ϕ_i , $i = 1, \dots, n$ are *holomorphic*.

Proposition

Let M^m be a complex manifold and $f : M^m \rightarrow \mathbb{R}^n$ be a pluri-conformal immersion. Then the Riemannian metric induced on M is *Kähler* if and only if the vectors $\frac{\partial}{\partial \bar{z}_j} \phi_k$ are *normal*. Consequently if f is pluri-minimal the induced metric is Kähler.

The algebra \mathcal{C} of **para-complex numbers** is the real algebra generated by 1 and the para-complex unit e , such that $e^2 = 1$. For all $z = x + ey$ one define

1) The **para-complex conjugation** :

$$\begin{aligned}\bar{\cdot} : \mathcal{C} &\rightarrow \mathcal{C} \\ x + ey &\mapsto x - ey\end{aligned}$$

2) **real** and **imaginary parts**

$$\Re(z) = \frac{z + \bar{z}}{2} = x, \quad \Im(z) = \frac{(z - \bar{z})e}{2} = y$$

Remark : $\mathcal{C} \cong \mathbb{R} \oplus \mathbb{R} \cong \mathcal{C}l_{0,1}$.

Definition

Let V a finite dimensional real vector space. A **para-complex structure** is an endomorphism $J \in \text{End}(V)$, such that

$$J^2 = \text{Id}, \quad \dim V^+ = \dim V^-,$$

with $V^\pm = \text{Ker}(\text{Id} \mp J)$.

We call (V, J) a **para-complex vector space**.

Definition

An **almost para-complex structure** on a manifold M is an endomorphism field $J \in \Gamma(\text{End}(TM))$ such that, for all $p \in M$, J_p is a para-complex structure on $T_p M$.

J is **integrable** iff the distributions $T^\pm M := \ker(\text{Id} \mp J)$ are integrable.

An **para-complex structure** on M is an integrable almost para-complex structure on M . We call (M, J) **para-complex manifold**.

$$\dim_{\mathbb{C}} M = \frac{\dim M}{2}.$$

Let (M, J) be a para-complex manifold. $TM = TM^+ \oplus TM^-$.

1. "adapted" coordinates : Frobenius

\Rightarrow There exists an open neighborhood U of M , and functions z_{\pm}^i , $i = 1, \dots, n$ on U

- which are constant on the leaves of TM^{\mp} ,
- whose differential dz_{\pm}^i are linearly independent.

$\Rightarrow (z_+^1, \dots, z_+^n, z_-^1, \dots, z_-^n)$ is a local coordinates system.

2. Let

$$x_i = \frac{z_+^i + z_-^i}{2}, \quad y_i = \frac{z_+^i - z_-^i}{2}.$$

It is a system of real local coordinates on U .

3. Para-holomorphic coordinates :

Definition

A smooth map $\phi : (M, J_M) \rightarrow (N, J_N)$ is para-holomorphic iff $d\phi J_M = J_N d\phi$.

A para-holomorphic local coordinates system is a system of para-holomorphic functions z^i , $i = 1, \dots, n$ on $U \subset M$, such that $x_i = \Re(z^i)$, $y_i = \Im(z^i)$.

- **$(p+, q-)$ -forms** : The decomposition $TM = TM^+ \oplus TM^-$ extends to

$$\Lambda^k T^*M = \bigoplus_{k=p+q} \Lambda^{p+, q-} T^*M.$$

and to the differential forms on M :

$$\Omega^k(M) = \bigoplus_{k=p+q} \Omega^{p+, q-}(M).$$

$$d : \Omega^k(M) \rightarrow \Omega^{k+1},$$

$$\partial^+ : \Omega^{p+, q-}(M) \rightarrow \Omega^{(p+1)+, q-}, \quad \partial^- : \Omega^{p+, q-}(M) \rightarrow \Omega^{p+, (q+1)-}.$$

- **(p, q) -forms** : $TM^C := TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$, avec

$$T_p M^{1,0} := \{X + eJX | X \in T_p M\}, \quad T_p M^{0,1} := \{X - eJX | X \in T_p M\},$$

$$\Rightarrow \Lambda^k T^*M^C = \bigoplus_{k=p+q} \Lambda^{p, q} T^*M, \quad \Omega^k_C(M) = \bigoplus_{k=p+q} \Omega^{p, q}(M).$$

Pseudo-riemannian surfaces in $\mathbb{R}^{2,1}$

Para-complex Weierstraß representations

Theorem (L.)

Let M be a Lorentz surface. The following conditions are equivalent :

- 1) $F : M \rightarrow \mathbb{R}^{2,1}$ is a conformal immersion .
- 2) There exists a triple $\omega_+ = (\omega_{1+}, \omega_{2+}, \omega_{3+})$ of $(1+, 0-)$ -forms and a triple $\omega_- = (\omega_{1-}, \omega_{2-}, \omega_{3-})$ of $(0+, 1-)$ -forms on M such that
 - (i) $\omega_{1+}^2 + \omega_{2+}^2 - \omega_{3+}^2 = 0, \quad \omega_{1-}^2 + \omega_{2-}^2 - \omega_{3-}^2 = 0,$
 - (ii) $\omega_{1+}\omega_{1-} + \omega_{2+}\omega_{2-} - \omega_{3+}\omega_{3-} > 0.$
 - (iii) The forms ω_{i+} resp. ω_{i-} are ∂_+ -exact resp. ∂_- -exact.

satisfying equation

$$F(q) = \int_p^q (\omega_{1+} + \omega_{1-}, \omega_{2+} + \omega_{2-}, \omega_{3+} + \omega_{3-}) + \text{constant}.$$

- M Lorentzian surface in $\mathbb{R}^{2,1}$:
 - Konderak '01 : Para-complex Weierstraß representation for minimal Lorentz surfaces,
 - L. '06 : Real and para-complex Weierstrass representation for general Lorentz surfaces
⇒ spinorial representation,
 - Analogs of Enneper surface, Catenoid and helicoid (associated families).
- Generalization for higher dimension manifolds of split signature.