Gluing infinitely many minimal surfaces together

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12 novembre 2010

1. Breaking periodicity of periodic minimal surfaces.
2. Non compact Riemann surfaces.
3. Balancing and discrete analysis on graphs.

1. Breaking periodicity of periodic minimal surfaces

Example 1 : Adding handles to Riemann examples (w/ F. Morabito)

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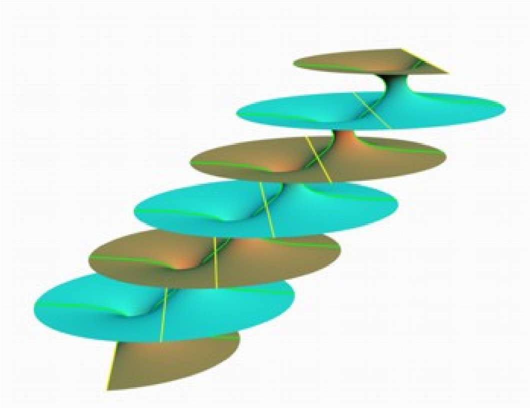
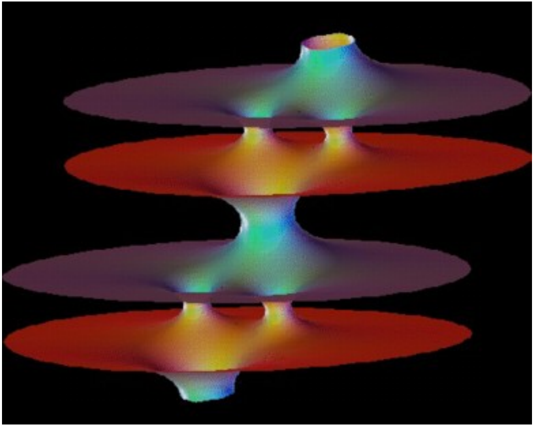
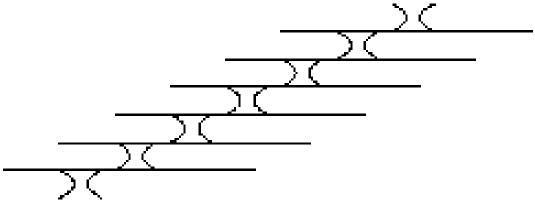
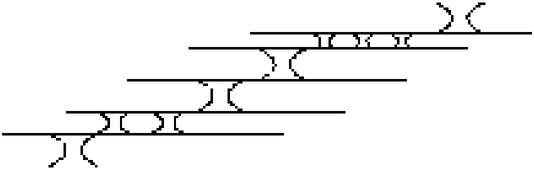


Riemann minimal example
(picture by Matthias Weber)



Riemann example with handles (F. Wei 90's)
(picture by Matthias Weber)





$$\cdots 1, 2, 1, 1, 3, 1, \cdots$$

n_k = number of necks at level k, $k \in \mathbb{Z}$.

Claim

It works provided

- $(n_k)_{k \in \mathbb{Z}}$ *is bounded*
- $\forall k \in \mathbb{Z}$, $(n_k - 1)(n_{k+1} - 1) = 0$

Example 2 : flips on Schwartz H surface

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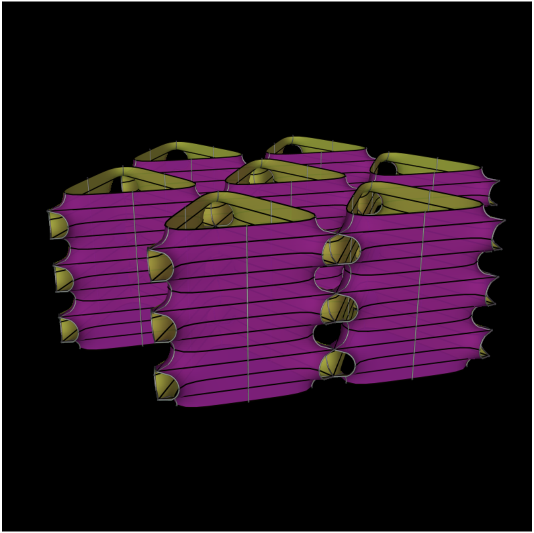
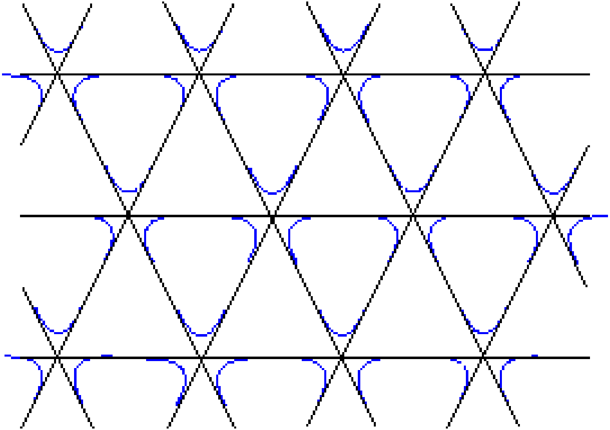
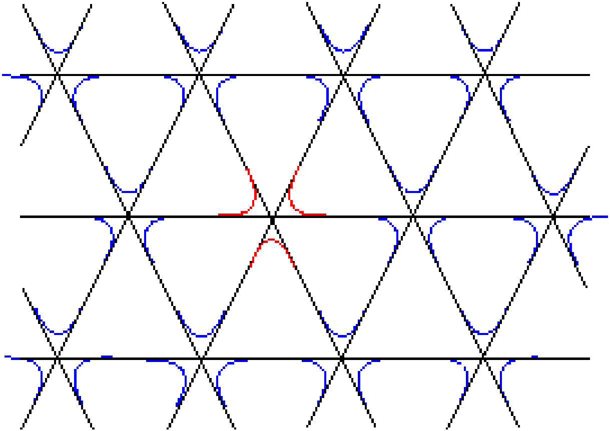


Schwartz triply periodic *H*-surface
(picture by Matthias Weber)

Construction of the *H* surface by desingularisation (R. Younes (2009) for the periodic case)



Breaking horizontal periodicity



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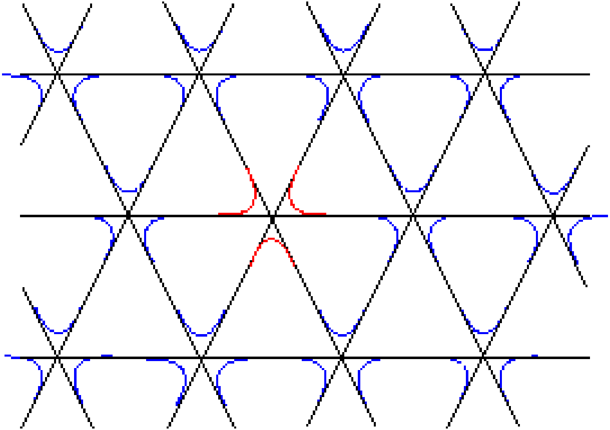

Claim

I can flip at a finite number of vertices.

2. Non-compact Riemann surfaces

Weierstrass representation

$$\psi(z) = \mathrm{Re} \int_{z_0}^{z} (\phi_1, \phi_2, \phi_3)$$

ϕ_1, ϕ_2, ϕ_3 : holomorphic 1-forms on a Riemann surface Σ

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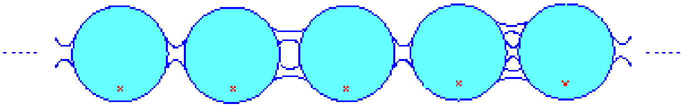
$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$$

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0$$

$$\mathrm{Re} \int_\gamma \phi_i = 0 \quad \forall \gamma \in H_1(\Sigma)$$

Conformal model for Riemann with handles



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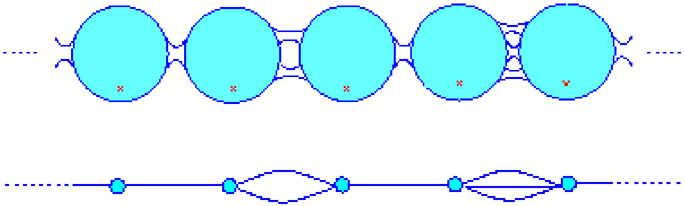
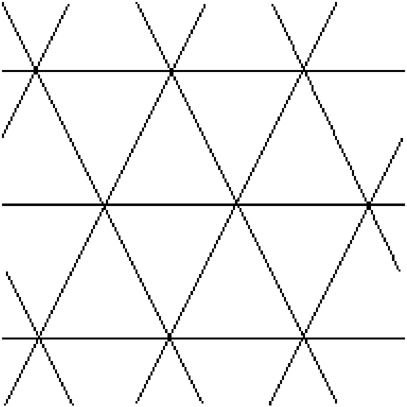


Conformal model for Schwartz H-surface



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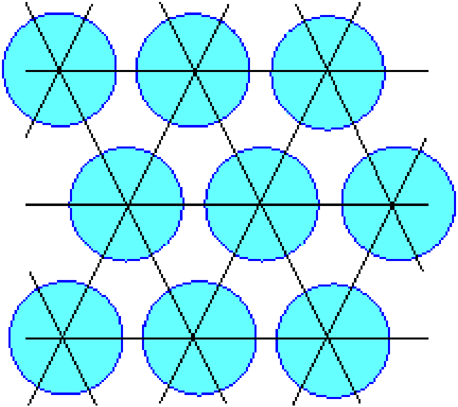

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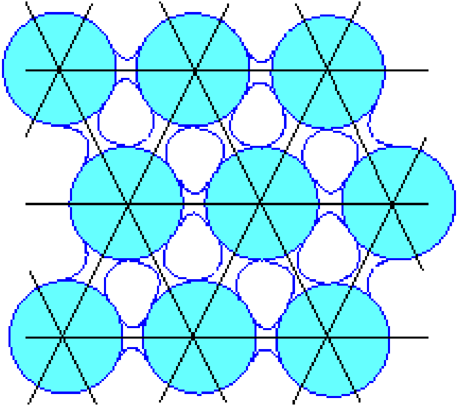

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Remove the disks $|z - p_e^-| < \sqrt{|t_e|}$ and $|z - p_e^+| < \sqrt{|t_e|}$.

Identify the points z and z' on the boundary circles such that

$$(z - p_e^-)(z' - p_e^+) = t_e$$

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If $t_e \neq 0$ this creates a neck connecting $\overline{\mathbb{C}}_v$ and $\overline{\mathbb{C}}_{v'}$.
If $t_e = 0$ this identifies p_e^- and p_e^+ and creates a node.
This defines a Riemann surface Σ possibly with nodes.

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Necessary condition (Cauchy theorem in $\overline{\mathbb{C}}_v$)

$$\forall v \in V, \sum_{e \in E_v^+} \alpha_e = \sum_{e \in E_v^-} \alpha_e \qquad (1)$$

E_v^- : edges which start at v
E_v^+ : edges which end at v
$E_v = E_v^- \cup E_v^+$

Finite case : Γ finite graph.

Theorem (Fay)

$\omega \mapsto (\int_{\gamma_e} \omega)_{e \in E}$ is an isomorphism from $\Omega^1(\Sigma)$ to the set of vectors $(\alpha_e)_{e \in E} \in \mathbb{C}^E$ which satisfy (1).

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Definition (Bers)

A differential ω is regular if it is holomorphic away from the nodes and for each node, it has simple poles at p_e^- and p_e^+, with opposite residues.

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Examples of admissible norms

- ℓ^p norms, $1 \leq p \leq \infty$
- weighted ℓ^p norm with weight w satisfying $\dfrac{w(v)}{w(v')} \leq c$

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Define norms

$$||\alpha|| = N\left((\sum_{e\in E_v} |\alpha_e|)_{v\in V} \right)$$

$$||\omega|| = N\left((\sup_{z\in\Omega_v} \left|\frac{\omega(z)}{dz}\right|)_{v\in V} \right)$$

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Theorem (T)

$\omega \mapsto (\int_{\gamma_e} \omega)_{e\in E}$ is an isomorphism of Banach spaces from the space of regular differentials ω with finite norm to the space of sequences $(\alpha_e)_{e\in E}$ with finite norm and satisfying (1).

3. Balancing and discrete analysis on graphs

Balancing for Riemann example with handles

Configuration : $p_{k,i} \in \mathbb{C}$, for $k \in \mathbb{Z}$,$1 \le i \le n_k$

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Forces :

$$F_{k,i} = 2 \sum_{\substack{j=1 \\ j \neq i}}^{n_k} \frac{c_k^2}{p_{k,i} - p_{k,j}} - \sum_{j=1}^{n_{k-1}} \frac{c_k c_{k-1}}{p_{k,i} - p_{k-1,j}} - \sum_{j=1}^{n_{k+1}} \frac{c_k c_{k+1}}{p_{k,i} - p_{k+1,j}}$$

with $c_k = \frac{1}{n_k}$.

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Definition

The configuration $(p_{k,i})_{k \in \mathbb{Z}, 1 \leq i \leq n_k}$ is balanced if $F_{k,i} = 0$ for all k, i.

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$dF = -\Delta$ (discrete Laplacian on $\mathbb{Z}$)

Problem : $\Delta : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ neither injective nor surjective.

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Conclusion : we can use the ℓ^∞ norm for this problem.

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Linearised operator

Perturb Γ by a function $h : V \to \mathbb{R}^2$, namely $v(t) = v + th_v$

$$L_v = \frac{d}{dt} F_v(t)|_{t=0}$$

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Question : find norms so that L is an invertible operator.

Discrete Laplacian on $\mathbb{Z}^d$

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Consider a function $u : \mathbb{Z}^d \to \mathbb{R}$

- $D_i u(x) = u(x + e_i) - u(x)$ denotes its discrete derivative in direction e_i
- D denotes any 1st order discrete derivative
- D^k denotes any k-th order discrete derivative
- $\Delta u : \mathbb{Z}^d \to \mathbb{R}$ denotes its discrete Laplacian

$$\Delta u(x) = \sum_{i=1}^{d} u(x + e_i) + u(x - e_i) - 2u(x)$$

Discrete weighted Sobolev spaces

Consider the weight $w(x) = 1 + |x|$.

$$||u||_{\ell^p_\beta} = \left(\sum_{x \in \mathbb{Z}^d} |u(x)|^p w(x)^{\beta p} \right)^{1/p}$$

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If $d \geq 3$, $1 < p < \infty$ and $2 - \frac{d}{p} < \beta < d - \frac{d}{p}$, then

$$\Delta : W^{2,p}_{\beta-2}(\mathbb{Z}^d) \to \ell^p_\beta(\mathbb{Z}^d)$$

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Discrete version of same result for $\mathbb{R}^d$ (Bartnik...)

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Conclusion : we can use weighted discrete Sobolev spaces for this problem.