Convex domains of Finsler and Riemannian manifolds

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Outline

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Convexity, Riemannian setting

Let (M, g_R) be a Riemannian manifold.

- Hopf-Rinow theorem: metric (or equivalently geodesic) completeness of M is a sufficient condition for convexity.
- (M, g_R) is convex if each two points of M can be joined by a non-necessarily unique geodesic which minimizes the distance in M.
- If D is an open domain of M (i.e. an open connected subset of M) with differentiable boundary ∂D: if M is complete, the convexity of D is equivalent to the convexity of ∂D.

  R. Bartolo, A.V.G., M. Sánchez, Ann. Global Anal. Geom., (2002)

Convexity of the boundary

Let D be an open domain of (M, g_R) with diff. boundary ∂D.

- ∂D is infinitesimally convex at $x \in \partial D$ ($(IC)_x$) if the second fundamental form σ_x with respect to the interior normal is positive semidefinite.

  A characterization: a differentiable function $\phi : M \to \mathbb{R}$ exists such that

$$\begin{cases} \phi^{-1}(0) = \partial D & \\ \phi > 0 & \text{on } D \\ \mathrm{d}\phi(x) \neq 0 & \text{for every } x \in \partial D \end{cases} \tag{1}$$

  Then, ∂D is infinitesimally convex at $x \in \partial D$ if and only if for one (and then for all) function ϕ satisfying (1):

$$H_\phi(x)[y, y] \leq 0 \quad \forall y \in T_x\partial D$$

  where $H_\phi(\cdot)[\cdot, \cdot]$ is the Hessian of ϕ.

- ∂D is locally convex at $x \in \partial D$ ($(LC)_x$) if a neighborhood $U \subset M$ of x exists such that

$$\exp_x (T_x \partial D) \cap (U \cap D) = \emptyset.$$

It is easy to prove that $(LC)_x \Rightarrow (IC)_x$.

- The converse is not true in general.
- When $(IC)_y$ holds for any $y \in U$, U neighborhood of x, then

$$(IC)_x \Rightarrow (LC)_x$$

  - when g_R is C^4 (R. L. Bishop, Indiana Math. J. 1974);
  - when g_R has constant curvature (M. P. Do Carmo, F. W. Warner, J. Differential Geom. 1970).

Global definitions

- ∂D is infinitesimally convex (IC) if it is $(IC)_x$ for any $x \in \partial D$;
- ∂D is locally convex (LC) if it is $(LC)_x$ for any $x \in \partial D$.

An "intermediate" notion:

- ∂D is geometrically convex (GC) if for any $p, q \in D$ the range of any geodesic $\gamma : [a, b] \to \overline{D}$ such that $\gamma(a) = p$ and $\gamma(b) = q$ satisfies

$$\gamma\left([a, b]\right) \subset D.$$

- It is easy to prove that

$$(LC) \Rightarrow (GC) \Rightarrow (IC).$$

- When g_R is C^4, by Bishop's theorem

$$(IC) \Rightarrow (LC)$$

thus (GC), (IC), (LC) are equivalent;

- A.V.G., Dynam. Systems Appl. (1995): when (M, g_R) is complete and ∂D is C^3,

$$(IC) \Rightarrow (GC)$$

Basic notions in Finsler Geometry

- D. Bao, S.S. Chern, Z. Shen, An Introduction to Riemann- Finsler Geometry. Graduate Texts in Mathematics. Springer-Verlag, New York (2000);

A Finsler structure on a smooth finite dimensional manifold M is a function $F\colon TM \to [0, +\infty)$ such that

- F is continuous on TM and smooth on $TM \setminus 0$;
- F is fiberwise positively homogeneous of degree one, i.e. $F(x, \lambda y) = \lambda F(x, y)$, for all $x \in M$, $y \in T_x M$ and $\lambda > 0$
- F^2 is fiberwise strongly convex i.e. the matrix

$$g_{i,j}(x,y) = \left[\frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x,y)\right] \tag{2}$$

is positive definite for any $(x, y) \in TM \setminus 0$.

Examples

- Each Riemannian manifold (M, g_R) is a Finsler one: $F(x, y) = \sqrt{g_R(x)[y, y]}$.
- Given a Riemannian manifold (M, h) and a one–form ω on M, a Randers metric on M is defined by

$$F(x, y) = \sqrt{h(x)[y, y]} + \omega(x)[y], \qquad \|\omega\|_x < 1,$$

where $\|\omega\|_x = \sup_{y \in T_x M \setminus \{0\}} |\omega(x)[y]| / \sqrt{h(x)[y, y]}$.

- The length of a piecewise smooth curve $\gamma\colon [a,b] \to M$ with respect to the Finsler structure F:

$$\ell_F(\gamma) = \int_a^b F(\gamma, \dot{\gamma})\,\mathrm{d}s.$$

- The distance between $p, q \in M$:

$$\mathrm{d}(p,q) = \inf_{\gamma \in \mathcal{P}(p,q;M)} \ell_F(\gamma),$$

where $\mathcal{P}(p,q;M)$ is the set of all piecewise smooth curves $\gamma\colon [a,b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$.
The distance function is non-negative and satisfies the triangle inequality, but it is not symmetric.

- For any $p \in M$ and $r > 0$ two different balls centered at p and having radius r:
  - the forward ball

$$B^+(p,r) = \{x \in M \mid \mathrm{d}(p,x) < r\}$$

  - the backward ball

$$B^-(p,r) = \{x \in M \mid \mathrm{d}(x,p) < r\}.$$

- a sequence $(x_n)_n \subset M$ is a forward (resp. backward) Cauchy sequence if for all $\varepsilon > 0$ there exists an index $\nu \in \mathbb{N}$ such that, for all $m \geq n \geq \nu$, it is $\mathrm{d}(x_n, x_m) < \varepsilon$ (resp. $\mathrm{d}(x_m, x_n) < \varepsilon$);
- a Finsler manifold is forward complete (resp. backward complete) if every forward (resp. backward) Cauchy sequence converges.

- Topologies induced by the forward balls and by the backward ones agree with the topology of M.
- Suitable versions of the Hopf-Rinow theorem hold stating the equivalence of forward (resp. backward) completeness and the compactness of closed and forward (resp. backward) bounded subsets of M.

If one of the above conditions holds, any pair of points in M can be joined by a geodesic minimizing the Finslerian distance, i.e. the (M, F) is convex.

Geodesics

Geodesics can be defined in different ways:

- using the Chern connection, geodesics equations become

$$\ddot{\gamma}^i(s) + \Gamma^i_{jk}(\gamma(s), \dot{\gamma}(s))\dot{\gamma}^j(s)\dot{\gamma}^k(s) = 0;$$

- geodesics are the critical points of the length functional;
- a smooth curve γ on $[a, b]$ is a geodesic parameterized with constant speed (i.e $s \mapsto F(\gamma(s), \dot{\gamma}(s)) = \mathrm{const.}$) if and only if it is a critical point of the energy functional

$$J(\gamma) = \frac{1}{2}\int_a^b F^2(\gamma, \dot{\gamma})\,\mathrm{d}s \qquad (3)$$

defined on the manifold of H^1 curves having fixed endpoints.

E. Caponio, M.A. Javaloyes, A. Masiello, Math. Ann., to appear

Convexity, Finslerian setting

Let D be an open domain of a Finsler manifold (M, F).

- ∂D is infinitesimally convex at x if the normal curvature with respect to the normal vector pointing into D is non-negative or (equivalently) if for a function ϕ as in the Riemannian case

$$H_\phi(x, y)[y, y] \leq 0 \quad \text{for every } y \in T_x\partial D,$$

where H_ϕ is the Finslerian Hessian of ϕ defined, for each $(x, y) \in TM \setminus 0$, as $H_\phi(x, y)[y, y] = \frac{\mathrm{d}^2}{\mathrm{d}s^2}(\phi \circ \gamma)(0)$, and γ is the geodesic of (M, F) (parameterized with constant speed) such that $\gamma(0) = x$ and $\dot{\gamma}(0) = y$.
Z. Shen, Lectures on Finsler geometry. World Scientific Publishing Co., Singapore (2001)

- Reversed Finsler metric $\tilde{F}$: for any $(x, y) \in TM$

$$\tilde{F}(x, y) = F(x, -y).$$

If γ is a geodesic on $[0, 1]$ of F, the reversed curve $\tilde{\gamma}(s) = \gamma(1 - s)$ in general is not a geodesic of F, but it is a geodesic for $\tilde{F}$.

- The notions of infinitesimal convexity for F and $\tilde{F}$ are equivalent.
- Two exponential maps:
  - $\exp$ associated to F;
  - $\widetilde{\exp}$ associated to $\tilde{F}$.

- ∂D is locally convex at $x \in \partial D$ if a neighborhood $U \subset M$ of x exists such that

$$\exp_x(T_x\partial D) \cap (U \cap D) = \emptyset \quad \widetilde{\exp}_x(T_x\partial D) \cap (U \cap D) = \emptyset.$$

- ∂D is geometrically convex if for any $p, q \in D$ the range of any geodesic $\gamma : [a, b] \to \overline{D}$ such that $\gamma(a) = p$ and $\gamma(b) = q$ satisfies

$$\gamma([a, b]) \subset D.$$

(D, F) is (GC) if and only if $(D, \tilde{F})$ is (GC).

Convexity of D and convexity of ∂D, Finslerian setting

- A.A. Borisenko, E.A Olin, Mathematical Notes, (2010): Bishop's technique only works for Berwald spaces.
- The relation between the convexity of D and ∂D is not clear (a strict notion of convexity as a technical assumption is necessary).
- For non-reversible Finsler metrics, there is no a priori a clear equivalent hypothesis to the completeness of $\overline{D}$.

In R. Bartolo, E. Caponio, A.V.G., M. Sánchez, Calc. Var. (to appear)

- the natural equivalence of the different convexities is proved;
- the correspondence between the convexity of D and that of ∂D is stated under a suitable completeness assumption.

Finslerian Bishop's Theorem

Theorem

Let (M, F) be a smooth Finsler manifold such that

- the fundamental tensor is $C^{1,1}_{\text{loc}}$ (i.e. its components are C^1 in $TM \setminus 0$ with locally Lipschitz derivatives)

and let $N \subset (M, F)$ be a $C^{2,1}_{\text{loc}}$ embedded hypersurface (i.e., N is locally regarded as the inverse image of some $C^{2,1}$ regular function).
Let $p \in N$ and choose a transverse direction as inner pointing in some neighborhood U of p.
If N is infinitesimally convex in $U \cap N$, then N is locally convex at p (and, thus, on all $U \cap N$).

- This result applies to Riemannian manifolds requiring $C^{1,1}$ differentiability.

Sketch of the proof

Denote by D the inner domain of N.

Proposition

Assume that ∂D is infinitesimally convex in a neighborhood U of $p \in \partial D$. Let $\gamma : [0, b] \to U$ be a geodesic which satisfies $\gamma(0) = p$, $\gamma(]0, b]) \subset U \cap D$. Then, $\dot\gamma(0) \notin T_p(\partial D)$.

Proof:

- If by contradiction that $\dot\gamma(0) \in T_p\partial D$, then $\gamma([0, \sigma[) \subset \partial D$ for some $\sigma > 0$.

It is possible to project γ on ∂D, obtaining a curve $\gamma_\Pi : [-\sigma, \sigma] \to \partial D$ such that (by infinitesimal convexity)

$$H_\phi(\gamma_\Pi(s), \dot\gamma_\Pi(s))[\dot\gamma_\Pi(s), \dot\gamma_\Pi(s)] \le 0, \quad \text{for every } s \in [-\sigma, \sigma].$$

Set $\rho(s) = \phi(\gamma(s))$, it follows $\ddot{\rho}(s) = H_\phi(\gamma(s), \dot{\gamma}(s))[\dot{\gamma}(s), \dot{\gamma}(s)]$.
Thus

$$\ddot{\rho}(s) \leq H_\phi(\gamma(s), \dot{\gamma}(s))[\dot{\gamma}(s), \dot{\gamma}(s)] - H_\phi(\gamma_\Pi(s), \dot{\gamma}_\Pi(s))[\dot{\gamma}_\Pi(s), \dot{\gamma}_\Pi(s)]$$

- As in local coordinates

$$(H_\phi)_{ij}(x, y)y^i y^j = \frac{\partial^2 \phi}{\partial x^i \partial x^j}(x)y^i y^j - \frac{\partial \phi}{\partial x^k}(x)\Gamma^k_{ij}(x, y)y^i y^j,$$

  using the regularity assumptions, we obtain

$$\ddot{\rho}(s) \leq C(\rho(s) + |\dot{\rho}(s)|).$$

- Thus

$$\begin{cases} \ddot{\rho}(s) \leq C(\rho(s) + |\dot{\rho}(s)|) \\ \rho(0) = 0, \dot{\rho}(0) = 0 \end{cases}$$

  which implies $\rho \equiv 0$.

Corollary

If ∂D is infinitesimally convex then ∂D is geometrically convex.

Proof: otherwise, a geodesic $\gamma : [0,1] \to \overline{D}$ with $\gamma(0), \gamma(1) \in D$ and $c \in]0,1[$ exist such that $\gamma(c) \in \partial D$ and $\gamma(]c,1]) \subset D$. Necessarily, $\dot{\gamma}(c) \in T_p \partial D$, which is a contradiction.

Proof of the Finslerian Bishop's Theorem

- If, by contradiction $N = \partial D$ is not locally convex at $p \in N$, a sequence of tangent vectors $v_n \in T_p\partial D$ exists such that $(v_n)_n \to 0$ and for any n

$$p_n = \exp(v_n) \in D \cap U$$

or for any n

$$q_n = \widetilde{\exp}(v_n) \in D \cap U.$$

- Assume that $\big(p_n = \exp(v_n)\big)_n$ exists such that $p_n \in D \cap U$ (the other case is similar).

- It is possible to fix a small enough convex ball $B^+(p, \delta)$ such that
  - $\partial D \cap B^+(p, \delta) \subset U$;
  - for each $q_1, q_2 \in D \cap B^+(p, \delta)$ the (unique) geodesic in $B^+(p, \delta)$ which connects q_1 with q_2 is included in D.

  Assume that for any n

$$p_n \in D \cap B^+(p, \delta).$$

- Each unit speed geodesic $\alpha_n : [0, b_n] \to U$ which connects p_n with p_1 is included in $D \cap B^+(p, \delta)$.

- The sequence $(\alpha_n)_n$ uniformly converges to the (unique, up to rep.) geodesic $\alpha : [0, b] \to \overline{D} \cap \overline{B}^+(p, \delta)$ connecting p with p_1.
- For some $t_0 \in [0, b[$ it is
  - $\alpha(t_0) \in \partial D$;
  - $\dot{\alpha}(t_0) \in T_{\alpha(t_0)}\partial D$;
  - for any $t \in]t_0, b[\ \alpha(t) \in D$;
- On the other hand, from previous proposition, it must be

$$\dot{\alpha}(t_0) \notin T_{\alpha(t_0)}\partial D.$$

Finslerian Bishop's Theorem and straightforward implications

$$(LC) \Rightarrow (GC) \Rightarrow (IC)$$

yields the full equivalences among the notions of convexity for the boundary of a domain D.

Corollary

Let D be a $C^{2,1}_{\mathrm{loc}}$ domain of a manifold M endowed with a Finsler metric whose fundamental tensor is $C^{1,1}_{\mathrm{loc}}$ on $TM \setminus 0$.
It is equivalent for ∂D to be:

- *infinitesimally convex;*
- *geometrically convex;*
- *locally convex.*

Convexity of ∂D and the convexity of D

- The symmetrized distance on M is defined by

$$\mathrm{d}_s(p,q) = \frac{1}{2}\left(\mathrm{d}(p,q) + \mathrm{d}(q,p)\right).$$

E. Caponio, M.A. Javaloyes, M. Sánchez, preprint (2009):
if for all $p \in M, r > 0$,

$$\overline{B}_s(p,r) \text{ is compact}$$

where

$$B_s(p,r) = \{x \in M \mid \mathrm{d}_s(x,p) < r\}$$

then

- the metric space (M, d_s) is complete;
- (M, F) is convex.

- E. Caponio, M.A. Javaloyes, A. Masiello, Math. Ann., to appear: if (M, F) forward or backward complete, then it is convex.

Theorem

Let D be a $C^{2,1}_{\mathrm{loc}}$ domain of a smooth Finsler manifold (M, F) with $C^{1,1}_{\mathrm{loc}}$ fundamental tensor such that

- *for any $p \in \overline{D}$ and $r > 0$*

$$\overline{B}_s(p, r) \cap \overline{D} \text{ is compact.}$$

Then, D is convex if and only if ∂D is convex.
Moreover, if D is not contractible, any pair of points in D can be joined by infinitely many connecting geodesics contained in D and having diverging lengths.

The variational setting

Let D be a $C^{2,1}_{\rm loc}$ domain of a Finsler manifold (M, F).

- for any $p, q \in D$,

$$\Omega(p,q;D) = \Big\{ \gamma : [0,1] \to D \mid \gamma \text{ abs. cont.} \\ \textstyle\int_0^1 h(\gamma)[\dot\gamma,\dot\gamma]\,\mathrm{d}s < +\infty, \gamma(0) = p, \gamma(1) = q \Big\}$$

  where h is any complete auxiliary Riemannian metric on M.

- Denoting the function F^2 by G,

$$J \colon \Omega(p,q;D) \to \mathbb{R}, \qquad J(\gamma) = \frac{1}{2}\int_0^1 G(\gamma,\dot\gamma)\,\mathrm{d}s.$$

- Critical points γ of J are all and only the geodesics of the Finsler manifold (M, F) connecting p to q and having support in D.
- J does not satisfy the Palais-Smale condition: Palais-Smale sequences may converge to curves touching ∂D.
- A functional $F : \Omega \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $(x_m)_m \subset \Omega$ such that

$$(F(x_m))_m \text{ is bounded and } \mathrm{d}F(x_m) \to 0$$

admits a converging subsequence.

The penalization method

- For any $\varepsilon \in]0,1]$, define on $\Omega(p,q;D)$ the functional

$$J_\varepsilon(\gamma) = J(\gamma) + \int_0^1 \frac{\varepsilon}{\phi^2(\gamma)}\,\mathrm{d}s.$$

- This penalization technique is due to W.B. Gordon, J. Differential Geom. (1974): the penalizing term becomes infinite close to the boundary.
- for any $\varepsilon \in]0,1]$ and for any $c \in \mathbb{R}$, the sublevels

$$J_\varepsilon^c = \{x \in \Omega(p,q;D) \mid J_\varepsilon(x) \leq c\}$$

  are complete metric subspaces of $\Omega(p,q;D)$;
- for any $\varepsilon \in]0,1]$, J_ϵ satisfies the Palais-Smale condition.

- A regularity result: for any $\varepsilon \in]0,1]$, let $\gamma_\varepsilon \in \Omega(p,q;D)$ be a critical point of J_ε. Then γ_ε is C^1 and it is C^2 in a neighborhood of any $\bar{s} \in [0,1]$ such that $\dot{\gamma}_\varepsilon(\bar{s}) \neq 0$, where (in local coordinates) it verifies

$$\ddot{\gamma}_\varepsilon^i(s) + \Gamma^i_{jk}(\gamma_\varepsilon(s), \dot{\gamma}_\varepsilon(s))\dot{\gamma}_\varepsilon^j(s)\dot{\gamma}_\varepsilon^k(s) = \\ -\frac{2\varepsilon}{\phi^3(\gamma_\varepsilon(s))}\partial_{x^k}\phi(\gamma_\varepsilon(s))g^{ki}(\gamma_\varepsilon(s), \dot{\gamma}_\varepsilon(s)).$$

- A constant $E_\varepsilon(\gamma_\varepsilon) \in \mathbb{R}$ exists such that

$$E_\varepsilon(\gamma_\varepsilon) = \frac{1}{2}G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) - \frac{\varepsilon}{\phi^2(\gamma_\varepsilon)} \quad \text{on } [0,1].$$

- Then, for any $\varepsilon \in]0,1]$, J_ε has a minimum point $\gamma_\varepsilon \in \Omega(p,q;D)$, then $k > 0$ exists such that

$$J_\varepsilon(\gamma_\varepsilon) \leq k \quad \text{for all } \varepsilon \in]0,1],$$

  since $J_\varepsilon(\gamma_\varepsilon) \leq J_\varepsilon(\gamma_1) \leq J_1(\gamma_1)$.

A priori estimates and limit process

- Setting

$$\lambda_\varepsilon(s) = \frac{2\varepsilon}{\phi^3(\gamma_\varepsilon(s))} \quad \text{for all } \varepsilon \in]0,1] \text{ and } s \in [0,1],$$

  $\varepsilon_0 \in]0,1]$ exists such that $(\|\lambda_\varepsilon\|_\infty)_{\varepsilon \in]0,\varepsilon_0]}$ is bounded, where $\|\lambda_\varepsilon\|_\infty = \max_{s\in[0,1]} \lambda_\varepsilon(s)$.
- A subsequence $(\varepsilon_m)_m$ in $]0,1]$ exists such that $(\gamma_{\varepsilon_m})_m$ strongly converges to a curve γ which is a geodesic joining p and q in D.

D is convex

- since J is a continuous functional, recalling that γ_{ε_m} is a minimum for J_{ε_m} and $(\gamma_{\varepsilon_m})_m$ converges to γ in $\Omega(p, q; D)$, it is

$$J(\gamma) = \lim_m J(\gamma_{\varepsilon_m}) \leq \lim_m J_{\varepsilon_m}(\gamma_{\varepsilon_m}) \leq \lim_m J_{\varepsilon_m}(\bar{\gamma}) = J(\bar{\gamma}),$$

  for any other curve $\bar{\gamma} \in \Omega(p, q; D)$.
  Hence γ is a minimum for J and therefore also for the length functional ℓ_F.

- The multiplicity of geodesics connecting the points p and q and having support contained in D, under the assumption that D is not contractible is a standard application of Lusternik-Schnirelman category theory.

  Given a topological space X the Lusternik-Schnirelman category of $A \subset X$, denoted by $\mathrm{cat}_X(A)$, is the minimum number of closed contractible subsets of X needed to cover A. By definition $\mathrm{cat}_X(A) = +\infty$ if the covering cannot be realized by a finite number of subsets.

Differences with the Riemannian setting

- The critical points of J_ε are C^1 curves having supports in D but they are continuously twice differentiable only on the open subset of the domain of parametrization where their velocity vector field is not zero.
- The analogous to the Nash isometric embedding theorem does not hold in general for a Finsler manifold.
- The regularity results and the passage to the limit in the penalization technique are more complicate than in the Riemannian case.