On the geometry of certain surfaces in homogeneous 3-spaces

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Canonical coordinates and principal directions

1. **The ambient space $\mathbb{M}^2(c) \times \mathbb{R}$**
   - Constant Angle Surfaces in $\mathbb{M}^2(c) \times \mathbb{R}$
2. **Surfaces in $\mathbb{S}^2 \times \mathbb{R}$**
3. **Surfaces in $\mathbb{H}^2 \times \mathbb{R}$**
   - Minkowski model of $\mathbb{H}^2$
   - Minimality and Flatness
4. **Surfaces in Euclidean space $\mathbb{E}^3$**
5. **C.A.S. in Sol**

The ambient space $\mathbb{M}^2(c) \times \mathbb{R}$

Space forms with constant sectional curvature c:

- $c = 1 \Rightarrow \mathbb{M}^2(c) = \mathbb{S}^2 \Rightarrow$ the ambient space $\mathbb{S}^2 \times \mathbb{R}$
- $c = -1 \Rightarrow \mathbb{M}^2(c) = \mathbb{H}^2 \Rightarrow$ the ambient space $\mathbb{H}^2 \times \mathbb{R}$
- $c = 0 \Rightarrow \mathbb{M}^2(c) = \mathbb{R}^2 \Rightarrow$ the ambient space $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$

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Problem 1: Constant Angle Surfaces

A problem studied until now consists of the classification and characterization of Constant Angle Surfaces (CAS) in different ambient spaces. A CAS is an orientable surface whose unit normal makes a constant angle, denoted by θ, with a fixed direction.

The complete classification:

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- F. Dillen, M.I.M., *Constant Angle Surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Bull. Braz. Math. Soc. **40** (1) (2009), 85–97.
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Problem 2: Canonical directions

When the ambient is of the form $\mathbb{M}^2 \times \mathbb{R}$, a favored direction is $\mathbb{R}$. It is known that for a constant angle surface in $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or in $\mathbb{H}^2 \times \mathbb{R}$, the projection of $\frac{\partial}{\partial t}$ (where t is the global parameter on $\mathbb{R}$) onto the tangent plane of the immersed surface, denoted by T, is a principal direction with the corresponding principal curvature identically zero.

Question

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First answer in $\mathbb{S}^2 \times \mathbb{R}$

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First answer in $\mathbb{S}^2 \times \mathbb{R}$

The characterization of surfaces with a principal direction:

Theorem (Dillen, Fastenakels, Van der Veken, 2009)

Let M be an immersed surface in $\mathbb{S}^2 \times \mathbb{R}$ and p a point of M for which $\theta(p) \neq \{0, \frac{\pi}{2}\}$. Then T is a principal direction if and only if M considered as a surface in $\mathbb{E}^4$ is normally flat.

First answer in $\mathbb{S}^2 \times \mathbb{R}$

Proposition (classification result) - Dillen, Fastenakels, Van der Veken, 2009

A surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ is a surface for which T is a principal direction if and only if the immersion F is (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) in the neighborhood of a point p where $\theta(p) \notin \{0, \frac{\pi}{2}\}$ given by

$$F : M \to \mathbb{S}^2 \times \mathbb{R} : \; (x, y) \mapsto (F_1(x, y), \; F_2(x, y), \; F_3(x, y), \; F_4(x))$$

with

$$F_j(x, y) = \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v)) dv$$

for $j = 1, 2, 3$ where $\phi'(x) = \cos(\theta(x))$, $F_4'(x) = \sin(\theta(x))$, $(\alpha_1, \; \alpha_2, \; \alpha_3)$ is a curve in $\mathbb{S}^2$ and $F_1^2 + F_2^2 + F_3^2 = 1$. Moreover, α_1, α_2, α_3,ψ and ϕ are related by

$$\begin{aligned} \alpha_j'(y) \;\; &= \;\; -\cos(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \cos(\psi(x) + \phi(v)) dv \\ &\quad - \sin(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v)) dv. \end{aligned}$$

General things in $\mathbb{H}^2 \times \mathbb{R}$

Notations:

- $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$ the Riemannian product of $(\mathbb{H}^2(-1), g_H)$ and $\mathbb{R}$
- $\widetilde{g} = g_H + dt^2$ the product metric, t the (global) coordinate on $\mathbb{R}$
- $\widetilde{\nabla}$ the Levi Civita connection of $\widetilde{g}$
- $\partial_t = \frac{\partial}{\partial t}$ the unit vector field tangent to the $\mathbb{R}$-direction
- $\widetilde{R}$ either the curvature tensor $\widetilde{R}(X, Y) = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$, or the

Riemann-Christoffel tensor on $\widetilde{M}$ defined by $\widetilde{R}(W, Z, X, Y) = \widetilde{g}(W, \widetilde{R}(X, Y)Z)$.

- $F : M \longrightarrow \widetilde{M}$ - isometric immersion ($\dim M = 2$)
- ξ - a unit normal vector to M, A - its shape operator
- $g = \widetilde{g}|_M$ - metric on M, ∇ - corresponding Levi Civita connection

(G) $\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$, h the second fundamental form of M
(W) $\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$

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Some useful formulas

Since $\partial_t := \frac{\partial}{\partial t}$ is of unit length, we decompose it as $\partial_t = T + \cos\theta\ \xi$

where

- T is the projection on $T(M)$ with $|T| = \sin\theta$ and
- θ is the angle function : $\cos\theta = \widetilde{g}(\partial_t, \xi)$.

(E.G.)

$$\begin{aligned} R(X,Y,Z,W) \ = \ & g(AX,W)g(AY,Z) - g(AX,Z)g(AY,W) - \\ & g(X,W)g(Y,Z) + g(X,Z)g(Y,W) + \\ & g(X,W)g(Y,T)g(Z,T) + g(Y,Z)g(X,T)g(W,T) - \\ & g(X,Z)g(Y,T)g(W,T) - g(Y,W)g(X,T)g(Z,T) \end{aligned}$$

(E.C.) $(\nabla_X A)\,Y - (\nabla_Y A)\,X = \cos\theta\ (g(X,T)Y - g(Y,T)X)$

Computing the Gaussian curvature K, from the equation of Gauss it follows

$$K = \det A - \cos^2\theta.$$

Knowing that any vector field $X \in T(M)$ can be decomposed as $X = X_H + g(X,T)\partial_t$ we get

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Proposition (Dillen, M., 2009)

Let X be an arbitrary tangent vector to M. Then we have

$$\nabla_X T = \cos\theta AX \tag{1}$$

$$X(\cos\theta) = -g(AX, T). \tag{2}$$

If $\theta = const.$, then (2) yields $g(AT, X) = 0$, $\forall\ X \in T(M)$. Hence:

- if $T = 0$ on M, then ∂_t is always normal, so $M \subseteq \mathbb{H}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$.
- if $T \neq 0$ then T is a principal direction with principal curvature 0.

Question

Study surfaces in $\mathbb{H}^2 \times \mathbb{R}$ such that T remains a principal direction but with the corresponding principal curvature different from 0.

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First answers

In the following we suppose that θ **is different from** 0 **and** $\frac{\pi}{2}$.

Proposition (Dillen, M., Nistor, to appear Taiwan. J. Math.)

If $\theta \neq 0, \frac{\pi}{2}$, then we can choose local coordinates (x, y) on the surface M isometrically immersed in $\widetilde{M}$ with ∂_x in the direction of T s.t.

$$g(x,y) = \frac{1}{\sin^2\theta} dx^2 + \beta^2(x,y) dy^2 \tag{3}$$

$$A = \begin{pmatrix} \theta_x \sin\theta & \theta_y \sin\theta \\ \frac{\theta_y}{\sin\theta\beta^2} & \frac{\sin^2\theta\beta_x}{\cos\theta\beta} \end{pmatrix} \tag{4}$$

and the functions θ and β are related by the PDE

$$\frac{\sin^2\theta}{\cos\theta}\frac{\beta_{xx}}{\beta} + \frac{\sin\theta\theta_x}{\cos^2\theta}\frac{\beta_x}{\beta} + \frac{\theta_y}{\sin\theta}\frac{\beta_y}{\beta^3} + \left(2\frac{\cos\theta\theta_y^2}{\sin^2\theta} - \frac{\theta_{yy}}{\sin\theta}\right)\frac{1}{\beta^2} - \cos\theta = 0. \tag{5}$$

An analogue result formulated for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ having T as principal direction, is the following

Proposition (Dillen, M., Nistor, 2009)

Let M be isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$ with T a principal direction. Then, we can choose the local coordinates $(x, \ y)$ such that ∂_x is in the direction of T,

$$g = dx^2 + \beta^2(x, \ y)dy^2 \tag{6}$$

$$A = \begin{pmatrix} \theta_x & 0 \\ 0 & \tan\theta \frac{\beta_x}{\beta} \end{pmatrix}. \tag{7}$$

Moreover, the functions θ and β are related by the PDE

$$\beta_{xx} + \tan\theta\theta_x\beta_x - \beta\cos^2\theta = 0 \tag{8}$$

and $\theta_y = 0$.

Canonical coordinates

Remark

For every two functions θ and β defined on a smooth simply connected surface M such that $\theta_y = 0$ and $\beta_{xx} + \tan\theta\theta_x\beta_x - \beta\cos^2\theta = 0$ for certain coordinates (x, y), we can construct an isometric immersion $F : M \to \mathbb{H}^2 \times \mathbb{R}$ with the shape operator (7) and such that it has a canonical principal direction.

Remark

Let M be an isometrically immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ such that T is a principal direction. Coordinates (x, y) on M such that ∂_x is collinear with T and the metric g has the form $g = dx^2 + \beta^2(x, y)dy^2$ will be called *canonical coordinates*. Of course, they are not unique. More precisely, if (x, y) and $(\overline{x}, \overline{y})$ are both canonical coordinates, then they are related by $\overline{x} = \pm\, x + c$ and $\overline{y} = \overline{y}(y)$, where c is a real constant.

Minkowski model of the hyperbolic plane $\mathbb{H}^2$

Models for the hyperbolic plane:

1. the Klein model
2. the Poincaré disk
3. the upper half plane $\mathbb{H}^+$
4. Minkowski model $\mathcal{H}$

$$\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid x_1^2 + x_2^2 - x_3^2 = -1, \; x_3 > 0\}$$

with Lorentzian metric

$$\langle \, , \, \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

having constant Gaussian curvature -1.

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Characterization theorem

In order to study under which conditions T is a canonical principal direction, we regard the surface M as a surface immersed in $\mathbb{R}^3_1 \times \mathbb{R}$ (also denoted $\mathbb{R}^4_1$) having codimension 2.
The metric on the ambient space is given by $\widetilde{g} = dx_1^2 + dx_2^2 - dx_3^2 + dt^2$.
M is given by the immersion $F : M \to \mathbb{R}^3_1 \times \mathbb{R}$, $F = (F_1,\ F_2,\ F_3,\ F_4)$.

Theorem (Dillen, M., Nistor, 2009)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$. T is a principal direction if and only if M is normally flat in $\mathbb{R}^4_1$.

▸ Proof.

Classification theorem - version 1

Theorem (Dillen, M., Nistor, 2009)
If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with $\theta \neq 0, \frac{\pi}{2}$, then T is a principal direction if and only if F is given, up to isometries of $\mathbb{H}^2 \times \mathbb{R}$, by

$$F(x,\ y) = (F_1(x,y),\ F_2(x,y),\ F_3(x,y),\ F_4(x))$$

with $F_j(x,y) = A_j(y)\sinh\phi(x) + B_j(y)\cosh\phi(x),\ j = \overline{1,3}$ and $F_4(x) = \int_0^x \sin\theta(\tau)d\tau$, where $\phi'(x) = \cos\theta$. The six functions A_j and B_j are found in one of the following cases

- **Case 1**.

$$\begin{aligned} A_j(y) &= \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{1j} \\ B_j(y) &= \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{2j} \\ H_j'(y) &= B_j(y)\sinh\psi(y) - A_j(y)\cosh\psi(y) \end{aligned}$$

- **Case 2.**

$$\begin{aligned} A_j(y) &= \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{1j} \\ B_j(y) &= \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{2j} \\ H_j'(y) &= -A_j(y)\sinh\psi(y) + B_j(y)\cosh\psi(y) \end{aligned}$$

- **Case 3.**

$$\begin{aligned} A_j(y) &= \pm\int_0^y H_j(\tau)d\tau + c_{1j} \\ B_j(y) &= \int_0^y H_j(\tau)d\tau + c_{2j} \\ H_j'(y) &= c_{2j} \mp c_{1j} \end{aligned}$$

where $H = (H_1, H_2, H_3)$ is a curve on the de Sitter space $\mathbb{S}^2_1$, ψ is a smooth function on M and $c_1 = (c_{11}, c_{12}, c_{13})$, $c_2 = (c_{21}, c_{22}, c_{23})$ are constant vectors.

Clasiffication theorem - version 2

Theorem (Dillen, M., Nistor, 2009)

If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with angle function $\theta \neq 0, \frac{\pi}{2}$, then T is a principal direction if and only if F is given locally, up to isometries of the ambient space by

$$F(x, y) = (A(y) \sinh \phi(x) + B(y) \cosh \phi(x), \chi(x))$$

where $A(y)$ is a regular curve in $\mathbb{S}^2_1$, $B(y)$ is a regular curve in $\mathbb{H}^2_1$, such that $\langle A, B \rangle = 0$, $A' || B'$ and where $(\phi(x), \chi(x))$ is a regular curve in $\mathbb{R}^2$. The angle function θ of M depends only on x and coincides with the angle function of the curve (ϕ, χ). In particular we can arc length reparametrize (ϕ, χ); then (x, y) are canonical coordinates and $\theta'(x) = \kappa(x)$, the curvature of (ϕ, χ).

□

Clasiffication theorem - version 3

Theorem (Dillen, M., Nistor, 2009)

Let $F : M \to \mathbb{H}^2 \times \mathbb{R}$ be an isometrically immersed surface M in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M has T as a principal direction if and only if F is given, up to rigid motions of the ambient space, either by

$$F(x,y) = \Big(f(y)\cosh\phi(x) + N_f(y)\sinh\phi(x), \chi(x)\Big) \tag{9}$$

where $f(y)$ is a regular curve in $\mathbb{H}^2_1$ and $N_f(y) = \frac{f(y)\boxtimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$ represents the normal of f. Moreover, (ϕ, χ) is a regular curve in $\mathbb{R}^2$ and the angle function θ of this curve is the same as the angle function of the surface parameterized by F. □

Examples

Now, we would like to give some examples of surfaces that can be retrieved from the classification theorem. Let us consider first $\psi(y) = 0$ for all y in **Case 1**, getting

$$A_j(y) = \int_0^y H_j(\tau)d\tau + c_{1j},\ B_j(y) = c_{2j},\ H_j'(y) = -\int_0^y H_j(\tau)d\tau - c_{1j}.$$

The parametrization F in this case is given by

Example (rotational surface)

$$F(x,y) = \Big(\sin y \sinh\Big(\int_0^x \cos\theta(\tau)d\tau\Big),\ \cos y \sinh\Big(\int_0^x \cos\theta(\tau)d\tau\Big),$$
$$\cosh\Big(\int_0^x \cos\theta(\tau)d\tau\Big),\ \int_0^x \sin\theta(\tau)d\tau\Big).$$

Examples

Concerning **Case 3** in classification theorem, let us choose for example $c_1 = (0,1,0)$, $c_2 = (0,0,1)$ and $c_3 = (1,0,0)$. The parametrization in this case is given by

Example

$$F(x,y) = \Big(A(y)\sinh\Big(\int_0^x \cos\theta(\tau)d\tau\Big) + B(y)\cosh\Big(\int_0^x \cos\theta(\tau)d\tau\Big),\ \int_0^x \sin\theta(\tau)d\tau\Big)$$

where $A(y) = \left(y,\ 1-\frac{y^2}{2},\ \frac{y^2}{2}\right)$ and $B(y) = \left(y,\ -\frac{y^2}{2},\ 1+\frac{y^2}{2}\right)$.

Examples

If $\theta(x) = x^2$, the surface is

Example

$$F(x,y) = \left(A(y)\sinh\left(\sqrt{\tfrac{\pi}{2}}\; C\left(\sqrt{\tfrac{2}{\pi}}\; x\right)\right) + B(y)\cosh\left(\sqrt{\tfrac{\pi}{2}}\; C\left(\sqrt{\tfrac{2}{\pi}}\; x\right)\right), \sqrt{\tfrac{\pi}{2}}\; S\left(\sqrt{\tfrac{2}{\pi}}\; x\right)\right)$$

where C and S are the traditional notations for the Fresnel integrals $C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt$ respectively $S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$. The curve involved in the classification theorem is given in this case by $(\phi(x), \chi(x)) = (C(x),\ S(x))$, known as *Cornu spiral*.

Minimality

Theorem (Dillen, M., Nistor, 2009)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M is minimal with T as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by $F : M \longrightarrow \mathbb{H}^2 \times \mathbb{R}$

$$F(x,y) = \left(\frac{b(x)}{\sqrt{1+c_1^2-c_2^2}}, \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}} \sinh y, \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}} \cosh y, \ \chi(x) \right) \quad (10.a)$$

$$F(x,y) = \left(\frac{\sqrt{a^2(x)+1}}{\sqrt{c_2^2-c_1^2-1}} \cos y, \frac{\sqrt{a^2(x)+1}}{\sqrt{c_2^2-c_1^2-1}} \sin y, \frac{b(x)}{\sqrt{c_2^2-c_1^2-1}}, \ \chi(x) \right) \quad (10.b)$$

$$F(x,y) = \left(b(x)\ y, \ \frac{b(x)}{2}\,(1-y^2) - \frac{1}{2b(x)}, \ \frac{b(x)}{2}\,(1+y^2) + \frac{1}{2b(x)}, \ \chi(x) \right) \quad (10.c)$$

Minimality

Theorem (cont.)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M is minimal with T as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by

where

$$\chi(x) = \int_0^x \frac{1}{\sqrt{a^2(\tau)+1}}\, d\tau$$

with $a(x) = c_1 \cosh x + c_2 \sinh x$, $b(x) = a'(x)$ *and* $c_1, c_2 \in \mathbb{R}$.

Minimality in short

Remark

Since

$$F(x,y) = (A(y)\sinh\phi(x) + B(y)\cosh\phi(x), \chi(x)),$$

in general, under minimality assumption the curve $(\phi(x), \ \chi(x))$ is determined up to $c_1, c_2 \in \mathbb{R}$ by $\theta = \arctan\left(\frac{1}{c_1\cosh x + c_2\sinh x}\right)$, since $\phi'(x) = \cos\theta$ and $\chi'(x) = \sin\theta$. Moreover, in each case of the previous theorem the curves A and B are given by

$$A(y) = (1,\ 0,\ 0) \qquad B(y) = (0,\ \sinh y,\ \cosh y)$$

$$A(y) = (\cos y,\ \sin y,\ 0) \qquad B(y) = (0,\ 0,\ 1)$$

$$A(y) = \left(y,\ 1 - \frac{y^2}{2},\ \frac{y^2}{2}\right) \quad B(y) = \left(y,\ -\frac{y^2}{2},\ 1 + \frac{y^2}{2}\right).$$

Flatness

Theorem (Dillen, M., Nistor, 2009)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M is flat with T as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by
$F : M \longrightarrow \mathbb{H}^2 \times \mathbb{R}$

$$F(x,\ y) = \left(\frac{x}{\sqrt{c+1}} \cos y,\ \frac{x}{\sqrt{c+1}} \sin y,\ \frac{\sqrt{x^2+c+1}}{\sqrt{c+1}},\ \chi(x) \right)$$

$$F(x,\ y) = \left(\frac{\sqrt{x^2+c+1}}{\sqrt{-c-1}},\ \frac{x}{\sqrt{-c-1}} \sinh y,\ \frac{x}{\sqrt{-c-1}} \cosh y,\ \chi(x) \right)$$

$$F(x,\ y) = \left(xy,\ \frac{x}{2}(1-y^2) - \frac{1}{2x},\ \frac{x}{2}(1+y^2) + \frac{1}{2x},\ \chi(x) \right)$$

where

$$\chi(x) = \int^x \frac{\sqrt{\tau^2+c}}{\sqrt{\tau^2+c+1}}\, d\tau,\ \ c \in \mathbb{R}.$$

The upper half plane model of $\mathbb{H}^2$

Models for the hyperbolic plane:

1. the Klein model
2. the Poincaré disk
3. the upper half plane $\mathbb{H}^+$
4. Minkowski model $\mathcal{H}$

$$\mathbb{H}^+ = \{(X, Y) \in \mathbb{R}^2 \mid \; Y > 0\}$$

with metric

$$\langle \, , \, \rangle = \frac{dX^2 + dY^2}{Y^2}$$

having constant Gaussian curvature -1.

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The upper half plane model of $\mathbb{H}^2$

Method 1: Use **Cayley transformations** from $\mathcal{H}$ to H^+

$$x_1 = \frac{X}{Y} \qquad X = \frac{x_1}{x_3 - x_2}$$

$$x_2 = \frac{X^2 + Y^2 - 1}{2Y} \qquad Y = \frac{1}{x_3 - x_2}.$$

$$x_3 = \frac{X^2 + Y^2 + 1}{2Y}.$$

Method 2: Analytical approach - **solving the problem in $\mathbb{H}^+$** and then showing the consistence of results with $\mathcal{H}$:

A.I. Nistor, *On some special surfaces in* $\mathbb{H}^+ \times \mathbb{R}$, preprint 2010.

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Surfaces in $\mathbb{E}^3$ - minimality

Proposition (M., Nistor 2009)

Let M be a **minimal** isometric immersion in $\mathbb{E}^3$. We can choose (x, y)-local coordinates on M such that ∂_x is in direction of T, the metric of the surface can be expressed as

$$g = \frac{1}{\sin^2\theta}(dx^2 + dy^2) \tag{12}$$

and the shape operator A in the basis $\{\partial_x,\ \partial_y\}$ has the following expression

$$A = \sin\theta \begin{pmatrix} \theta_x & \theta_y \\ \theta_y & -\theta_x \end{pmatrix}. \tag{13}$$

Moreover, the function $\log\left(\tan\frac{\theta}{2}\right)$ is **harmonic**.

Example

$$\log\left(\tan\frac{\theta}{2}\right) \text{ is harmonic} \iff \Delta\log(\tan\tfrac{\theta}{2}) = 0 \iff$$
$$\cos\theta(\theta_x^2+\theta_y^2) - \sin\theta(\theta_{xx}+\theta_{yy}) = 0.$$

Under assumption $\theta_x = c\theta_y$ one gets that

$$\theta = 2\arctan(e^{d(cx+y)+d_0})$$

gives a **minimal** surface in $\mathbb{E}^3$.
Moreover, for any **harmonic** function f on M,

$$\theta = 2\arctan(e^f)$$

gives a **minimal** surface in $\mathbb{E}^3$.

Canonical coordinates in $\mathbb{E}^3$

The characterization theorem:

Theorem (M., Nistor, 2009)

Let M be an isometrically immersed surface in $\mathbb{E}^3$. Let (x, y) be orthogonal coordinates on M such that T is collinear to ∂_x. Then, T is a principal direction on M everywhere if and only if $\theta_y = 0$.

Canonical coordinates in $\mathbb{E}^3$

The classification theorem:

Theorem (M., Nistor, 2009)

A surface M isometrically immersed in $\mathbb{E}^3$ with T a canonical principal direction is given (up to isometries of $\mathbb{E}^3$) by one of the following cases:

- **Case 1.**

$$r : M \to \mathbb{E}^3,\ r(x,\ y) = \left(\phi(x)(\cos y,\ \sin y) + \gamma(y),\ \int_0^x \sin\theta(\tau)d\tau\right)$$

where $$\gamma(y) = \left(-\int_0^y \psi(\tau)\sin\tau d\tau,\ \int_0^y \psi(\tau)\cos\tau d\tau\right)$$

- **Case 2. (Cylinders)**

$$r : M \to \mathbb{E}^3,\ r(x,\ y) = \left(\phi(x)\cos y_0, \phi(x)\sin y_0,\ \int_0^x \sin\theta(\tau)d\tau\right) + y\gamma_0$$

where $\gamma_0 = (-\sin y_0, \cos y_0, 0)$, $y_0 \in \mathbb{R}$, $\phi'(x) = \cos\theta$.

Canonical coordinates in $\mathbb{E}^3$ - minimality

Theorem (M., Nistor, 2009)

Let M be a surface isometrically immersed in $\mathbb{E}^3$. M is a minimal surface with T a principal direction if and only if the immersion is, up to isometries of the ambient space, given by

$r : M \to \mathbb{E}^3$

$$r(x,\ y) = \left(\sqrt{x^2 + c^2}(\cos y,\ \sin y),\ \ln\left(x + \sqrt{x^2 + c^2}\right)\right),\ c \in \mathbb{R}.$$

Remark

Moreover, we notice that this surface can be obtained rotating the catenary around the Oz-axis. Hence, we obtain that the only minimal surface in the Euclidean space with a canonical principal direction is the **catenoid**.

Canonical coordinates in $\mathbb{E}^3$ - flatness

Theorem (M., Nistor, 2009)

Let M be a surface isometrically immersed in $\mathbb{E}^3$. M is a flat surface with T a principal direction if and only if the immersion is, up to isometries of the ambient space, given by

$$r : M \to \mathbb{E}^3, \quad r(x,\ y) = \left(\phi(x)\cos y_0,\ \phi(x)\sin y_0,\ \int_0^x \sin\theta(\tau)d\tau\right) + y\gamma_0$$

where $\gamma_0 = \big(-\sin y_0, \cos y_0, 0\big),\ \ y_0 \in \mathbb{R}$.
Here $\phi(x)$ represents a primitive of $\cos\theta$.

Notice that this is **Case 2.** **(Cylinders)** from the classification theorem.

Go to Sol

Sketch of proof

Proof.

With the previous considerations, for any $X \in T(M)$ we compute

$$D^{\perp}_X \tilde{\xi} = -\cos\theta \langle X, T\rangle \xi \quad \text{which implies} \quad D^{\perp}_X \xi = \cos\theta \langle X, T\rangle \tilde{\xi}.$$

Since *Proposition* 7 holds, the metric is given by (3) and using the previous expressions one has

$$R^{\perp}(\partial_x,\ \partial_y)\xi = \sin\theta\theta_y\tilde{\xi} \qquad \text{and} \qquad R^{\perp}(\partial_x,\ \partial_y)\tilde{\xi} = -\sin\theta\theta_y\xi.$$

Taking into account that ξ and $\tilde{\xi}$ are unitary and $\sin\theta$ cannot vanish, we get from the expressions above that M is normally flat if and only if $\theta_y = 0$. On the other hand, T is a canonical principal direction if and only if $\theta_y = 0$. This follows from expression (4) of the Weingarten operator A. Hence we get the conclusion. □

back

The geometry of Sol_3

- Sol_3: simply connected homogeneous 3-dimensional manifold whose isometry group has dimension 3.
- It is one of the eight models of geometry of Thurston.
- As Riemannian manifold : $\mathbb{R}^3$ equipped with the metric

$$\tilde{g} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$$

- The group operation

$$(x, y, z) * (x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z')$$

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The geometry of Sol_3

- The following transformations

$$(x, y, z) \mapsto (y, -x, -z) \quad \text{and} \quad (x, y, z) \mapsto (-x, y, z)$$

span a group of isometries of (Sol_3, g).

- This group is isomorphic to the dihedral group (with 8 elements) D_4. It is, in fact, the complete group of isotropy:

$$(x, y, x) \longmapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c)$$

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M. Troyanov, L'horizon de SOL, Exposition. Math. 16 (1998), 441–479.

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The geometry of Sol_3

With respect to the metric $\tilde{g}$ an orthonormal basis of left-invariant vector fields is given by

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

The Levi Civita connection $\widetilde{\nabla}$ of Sol_3 with respect to $\{e_1, e_2, e_3\}$ is given by

$$\begin{array}{lll} \widetilde{\nabla}_{e_1} e_1 = -e_3 & \widetilde{\nabla}_{e_1} e_2 = 0 & \widetilde{\nabla}_{e_1} e_3 = \ \ e_1 \\ \widetilde{\nabla}_{e_2} e_1 = \ \ 0 & \widetilde{\nabla}_{e_2} e_2 = e_3 & \widetilde{\nabla}_{e_2} e_3 = -e_2 \\ \widetilde{\nabla}_{e_3} e_1 = \ \ 0 & \widetilde{\nabla}_{e_3} e_2 = 0 & \widetilde{\nabla}_{e_3} e_3 = \ \ 0. \end{array}$$

Motivation

Constant angle surfaces were recently studied in product spaces $\mathbb{Q}_\epsilon \times \mathbb{R}$. The angle is considered between the normal of the surface and $\mathbb{R}$.

It is known, for Sol_3, that $\mathcal{H}^1 = \{dy = 0\}$ and $\mathcal{H}^2 = \{dx = 0\}$ are totally geodesic foliations whose leaves are the hyperbolic plane.

On the other hand, for $\mathbb{Q}_\epsilon \times \mathbb{R}$, the foliation $\{dt = 0\}$ is totally geodesic too (t is the global parameter on $\mathbb{R}$). Trivial examples for constant angle surfaces in $\mathbb{Q}_\epsilon \times \mathbb{R}$ are furnished by totally geodesic surfaces $\mathbb{Q}_\epsilon \times \{t_0\}$.

Let us consider $\mathcal{H}^2$. It follows that the tangent plane to $\mathbb{H}^2$ (the leaf at each $x = x_0$) is spanned by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, while the unit normal is e_1. So, this surface corresponds to $\mathbb{Q}_{-1} \times \{t_0\}$, case in which the constant angle is 0.

An oriented surface M, isometrically immersed in Sol_3, is called *constant angle surface* if the angle between its normal and e_1 is constant in each point of the surface M.

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First computations

López, M. - 2010: arXiv:1004.3889v1 [math.DG]

Denote by $\theta \in [0, \pi)$ the angle between the unit normal N and e_1. Hence

$$\widetilde{g}(N, e_1) = \cos\theta.$$

Let T be the projection of e_1 on the tangent plane:

$$e_1 = T + \cos\theta N.$$

Case $\theta = 0$. Then $N = e_1$ and hence the surface M is isometric to the hyperbolic plane $\mathcal{H}^2 = \{dx = 0\}$.

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First computations

From now on $\theta \neq 0$

$AT = -\widetilde{g}(N, e_3)T$, hence T is a principal direction on the surface

Let $E_1 = \frac{1}{\sin\theta}\ T$. Consider E_2 tangent to M, orthogonal to E_1 and such that the basis $\{e_1, e_2, e_3\}$ and $\{E_1, E_2, N\}$ have the same orientation.

It follows that

$$\begin{cases} e_1 = \sin\theta\ E_1 + \cos\theta\ N \\ e_2 = \cos\alpha\cos\theta\ E_1 + \sin\alpha\ E_2 - \cos\alpha\sin\theta\ N \\ e_3 = -\sin\alpha\cos\theta\ E_1 + \cos\alpha\ E_2 + \sin\alpha\sin\theta\ N \end{cases}$$

First computations

From now on $\theta \neq 0$

$AT = -\widetilde{g}(N, e_3)T$, hence T is a principal direction on the surface

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It follows that

$$\begin{cases} e_1 = & \sin\theta\ E_1 & & & + & \cos\theta\ N \\ e_2 = & \cos\alpha\cos\theta\ E_1 & + & \sin\alpha\ E_2 & - & \cos\alpha\sin\theta\ N \\ e_3 = & -\sin\alpha\cos\theta\ E_1 & + & \cos\alpha\ E_2 & + & \sin\alpha\sin\theta\ N \end{cases}$$

First computations

Case $\theta = \frac{\pi}{2}$. In this case e_1 is tangent to M and $T = E_1$.

$$h(E_1, E_1) = -\sin\alpha\ N,\ \ h(E_1, E_2) = 0,\ \ h(E_2, E_2) = \sigma\ N$$

$$E_1(\alpha) = 0 \quad \text{and} \quad E_2(\alpha) = \sin\alpha - \sigma.$$

Remark

The surface M is minimal if and only if $\sigma = \sin\alpha$. Since E_1 and E_2 are linearly independent, it follows that α is constant. Moreover, M is totally geodesic if and only if $\alpha = 0$, case in which M coincides with $\mathcal{H}^1$.

First computations

Due the fact that the Lie brackets of E_1 and E_2 is $[E_1, E_2] = \cos\alpha\ E_1$, one can choose local coordinates u and v such that

$$E_2 = \frac{\partial}{\partial u} \quad \text{and} \quad E_1 = \beta(u, v)\ \frac{\partial}{\partial v}\ .$$

Denote by

$$F : U \subset \mathbb{R}^2 \longrightarrow M \hookrightarrow Sol_3 \quad (u, v) \longmapsto \big(F_1(u, v),\ F_2(u, v),\ F_3(u, v)\big)$$

the immersion of the surface M in Sol_3.

It follows

$$F_1(v) = \int^v \frac{1}{\rho(\tau)}\ d\tau$$
$$F_2(u) = \int^u \left(\sin\alpha(\tau) e^{\int^\tau \cos\alpha(s) ds}\right) d\tau$$
$$F_3(u) = \int^u \cos\alpha(\tau) d\tau.$$

First results

Changing the v parameter, one gets the following parametrization

$$F(u,v) = \Big(v,\ \phi(u),\ \chi(u)\Big)$$

which represents a cylinder over the plane curve $\gamma(u) = (0,\ \phi(u),\ \chi(u))$ where $\phi(u) = \int^u \left(\sin\alpha(\tau) e^{\int^\tau \cos\alpha(s)ds}\right) d\tau$ and $\chi(u) = \int^u \cos\alpha(\tau) d\tau$.

Notice that **the surface is the group product between the curve $v \mapsto (v,\ 0,\ 0)$ and the curve γ.**

First results

θ arbitrary: we distinguish some particular situations for α:

Case $\sin\alpha = 0$. Then $\cos\alpha = \pm 1$ and the principal curvature corresponding to the principal direction T vanishes. Straightforward computations yield $\theta = \frac{\pi}{2}$ case which was discussed before.

Case $\cos\alpha = 0$. Such surface is minimal.

Proposition

The surface M given by the parametrization

$$F(u,v) = \left(\tan\theta\ e^{u\cos\theta},\ v,\ -u\cos\theta\right)$$

is a constant angle surface in Sol_3.

This surface is a (group) product between the curve $v \mapsto (0,\ v,\ 0)$ and the plane curve $\gamma(u) = (\tan\theta\ e^{u\cos\theta},\ 0,\ -u\cos\theta)$.

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General situation

The matrix of the Weingarten operator A with respect to the basis $\{E_1, E_2\}$ has the following expression

$$A = \begin{pmatrix} -\sin\alpha\sin\theta & 0 \\ 0 & \sigma \end{pmatrix}$$

for a certain function $\sigma \in C^\infty(M)$.
Moreover, the Gauss formula yields

$$E_1(\alpha) = 2\cos\theta\cos\alpha \quad E_2(\alpha) = \sin\alpha - \frac{\sigma}{\sin\theta}$$

and the compatibility condition

$$(\nabla_{E_1} E_2 - \nabla_{E_2} E_1)(\alpha) = [E_1, E_2](\alpha) = E_1(E_2(\alpha)) - E_2(E_1(\alpha))$$

gives rise to the following differential equation

$$E_1(\sigma) + \sigma\cos\theta\sin\alpha + \sigma^2\cot\theta = 2\sin\theta\cos\theta\sin^2\alpha.$$

Difficult computations

coordinate u such that $\frac{\partial}{\partial u} = E_1$.

$$\partial_u \alpha = 2\cos\theta\cos\alpha.$$

Solving this PDE one gets

$$\sin\alpha = \tanh(2u\cos\theta + \psi(v))$$

take v in such way that $\frac{\partial\alpha}{\partial v} = 0$, namely ψ is a constant

Denote: $I(u) = \int^u \sqrt{\cosh(2\tau\cos\theta + \psi_0)}d\tau$,

$J(u) = \int^u \cosh^{-\frac{3}{2}}(2\tau\cos\theta + \psi_0)d\tau$

Classification result

Theorem (López, M., 2010)

A general constant angle surface in Sol_3 can be parameterized as

$$F(u,v) = \gamma_1(v) * \gamma_2(u)$$

where

$$\gamma_1(v) = \left(\sin\theta \int^v \xi(\tau) e^{-\zeta(\tau)} d\tau, \ \pm\cos\theta \int^v \xi(\tau) e^{\zeta(\tau)} d\tau, \ \zeta(v) \right)$$

$$\gamma_2(u) = \left(\sin\theta \ I(u), \ \pm\cos\theta \ J(u), \ -\frac{1}{2} \log\cosh \bar{u} \right)$$

and ζ, ξ are arbitrary functions depending on v.

The curve γ_2 is parametrized by arclength.

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