The geometric Cauchy problem for surfaces associated to integrable systems

David Brander

Technical University of Denmark

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Outline

Outline

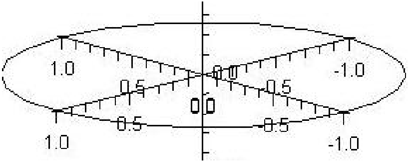
Background - Björlings Problem
- Björling's Problem

Special Submanifolds and Loop Group Methods
- Moving Frames and the Maurer-Cartan Form
- The spectral parameter and loop groups

Solution of the Björling Problem for CMC Surfaces
- The loop group formulation
- Solving the Björling problem
- Other related work

Problem: *Given a space curve C, find a mean curvature zero surface which contains this curve.*

Example: C **is a circle:**



Problem: *Given a space curve C, find a minimal surface which contains this curve.*

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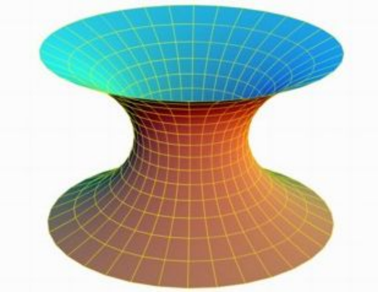

Figure: Catenoid

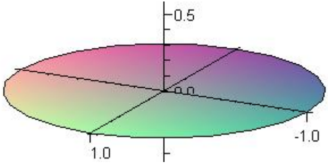

Figure: Unit disc

- Solution certainly not unique
- What other data can we specify to get a unique solution?

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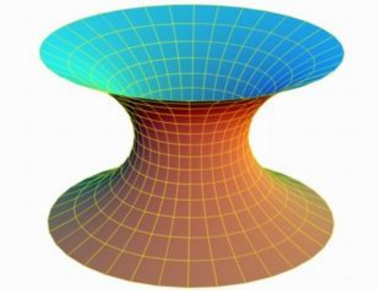

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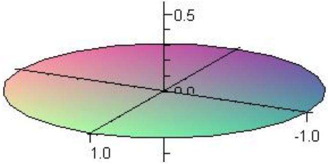

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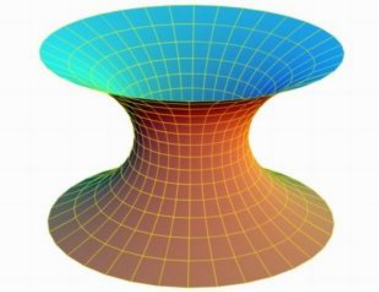

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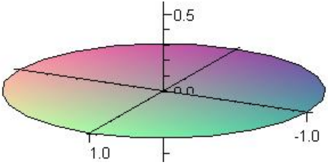

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The Björling Problem

Problem: Take a *real analytic* curve C, given by $\alpha : I \to \mathbb{R}^3$, and a real analytic family of tangent planes along C. Find a minimal surface containing C, whose tangent space along C is given by the family.

- Posed by EG Björling in 1844.
- Solution by H.A. Schwarz in 1890.
- Solution given by a **formula:**

$$f(z) = \Re\left\{\alpha(z) - i\int_{x_0}^{z} N(w) \times \alpha'(w)\mathrm{d}w\right\},$$

  - $\alpha(z)$ *holomorphic extension* of $\alpha(t)$
  - N unit normal along α, $N(z)$ holo. extension.

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The Weierstrass representation for minimal surfaces

The solution for Björling's problem can be understood this way:

- The Gauss map of a minimal surface is *holomorphic*.
- The **Weierstrass representation** gives a formula for the surface in terms of holomorphic data.
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The Björling problem for other types of surfaces

- Has been sudied for other classes of surfaces which have a (holomorphic) Weistrass representation (e.g. recent work by Jose Galvez, Pablo Mira and collobarators).
- Called the **geometric Cauchy problem:**
  Given a (real analytic) curve C, and a (real analytic) family of tangent planes along C. Find a (unique?) surface containing C, whose tangent space along C is given by the family.

Non-minimal Constant Mean Curvature Surfaces

- How about CMC H surfaces, $H \neq 0$?
- The Plateau problem had been studied (last half of 20th C.), but not the Björling problem.
- The Gauss map is not holomorphic.
- However, the Gauss map is *harmonic*,
- Harmonic maps have a representation as a *holomorphic* map into a loop group.

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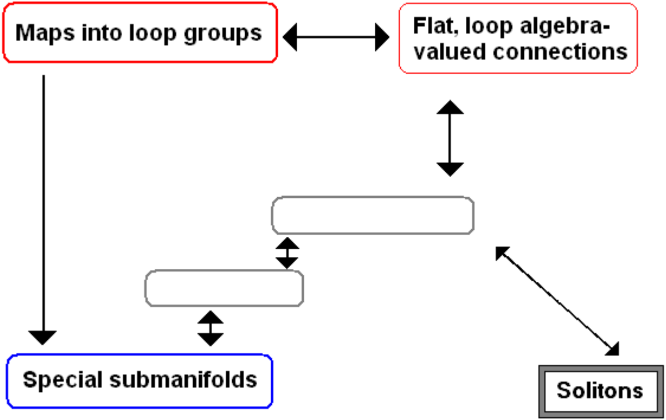
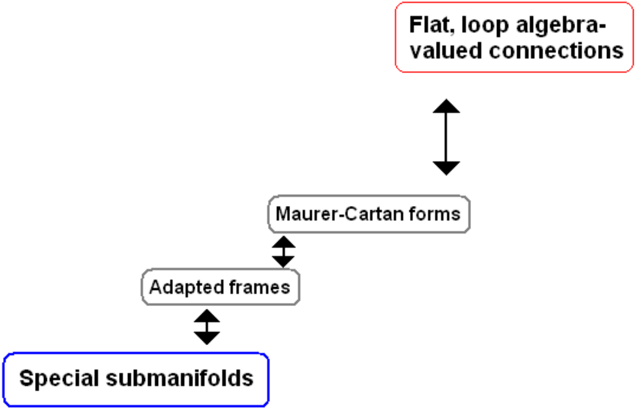
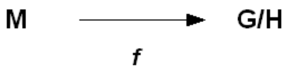
Special Submanifolds and Loop Group Methods




Flat, loop algebra-valued connections
Maurer-Cartan forms
Adapted frames
Special submanifolds

Moving Frames

- $f : M \to G/H$, immersed submanifold of a homogeneous space.



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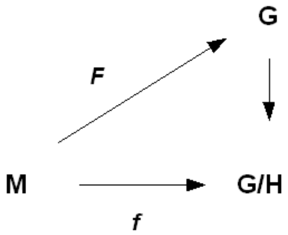

- Lift, $F : M \to G$, a **frame** for f.
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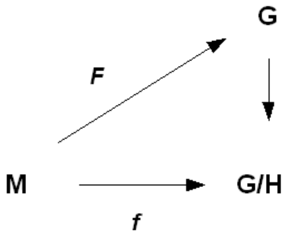

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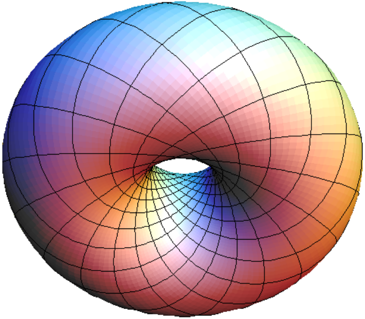

Figure: Clifford torus, $S^1 \times S^1 \subset S^3$, stereographically project to $\mathbb{R}^3$

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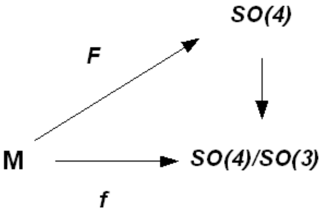
- **Adapted frame:**
  $F : \mathbb{R}^2 \to SO(4)$,

$$F := \begin{pmatrix} e_1 & e_2 & n & f \end{pmatrix},$$

  e_i tangent.



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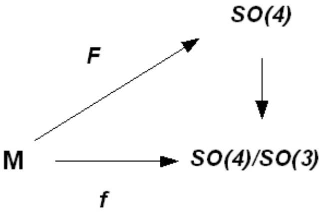

The Maurer-Cartan Form

Given a frame $F : M \to G$, for $f : M \to G/H$,

- **Maurer-Cartan form**, $\alpha = F^{-1}\mathrm{d}F \in \mathfrak{g} \otimes \Omega(M)$
- Satisfies the **Maurer-Cartan equation**

$$\mathrm{d}\alpha + \alpha \wedge \alpha = 0. \tag{1}$$

- **Converse:** if $\alpha \in \mathfrak{g} \otimes \Omega(M)$, satisfies (1) $\Rightarrow$ integrate to obtain $F : M \to G$.
- **Fundamental point:** α contains all geometric information about f.

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- **Adapted frame:** $F : \mathbb{R}^2 \to SO(4)$,

$$F := \begin{pmatrix} e_1 & e_2 & n & f \end{pmatrix}, \qquad e_i \text{ tangent.}$$

- **Maurer-Cartan form:**

$$\alpha = F^{-1}\mathrm{d}F = \begin{pmatrix} e_1^T \\ e_2^T \\ n^T \\ f^T \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}e_1 & \mathrm{d}e_2 & \mathrm{d}n & \mathrm{d}f \end{pmatrix}$$

$$= \begin{pmatrix} \omega & \beta & \theta \\ -\beta^t & 0 & 0 \\ -\theta^t & 0 & 0 \end{pmatrix},$$

- **Integrability:** $\mathrm{d}\alpha + \alpha \wedge \alpha = 0 \Leftrightarrow$

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$$\mathrm{d}\beta + \omega \wedge \beta = 0, \tag{3}$$

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- **Flatness:** $\mathrm{d}\omega + \omega \wedge \omega = 0.$
- Set

$$\alpha_\lambda = \begin{pmatrix} \omega & \lambda\beta & \lambda\theta \\ -\lambda\beta^t & 0 & 0 \\ -\lambda\theta^t & 0 & 0 \end{pmatrix} = a_0 + a_1\lambda.$$

- Then $\mathrm{d}\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0 \Leftrightarrow$
  $\mathrm{d}\omega + \omega \wedge \omega - \lambda^2(\beta \wedge \beta^t + \theta \wedge \theta^t) = 0$, plus (3) and (4).
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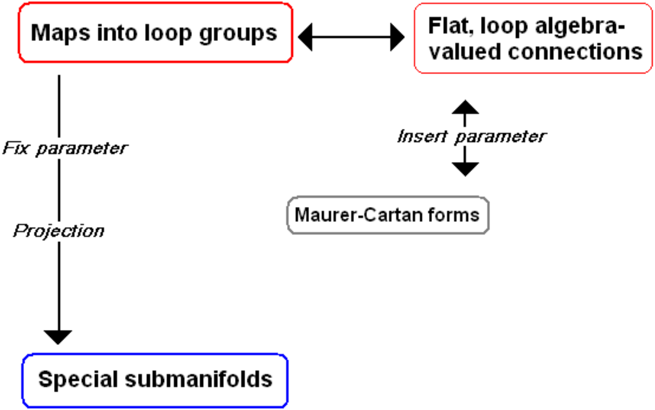
Maps into loop groups
Flat, loop algebra-valued connections
Fix parameter
Projection
Insert parameter
Maurer-Cartan forms
Special submanifolds

Parameterised Families of Frames

- for $\lambda \in \mathbb{C}^*$, 1-parameter *family* of 1-forms, $\alpha_\lambda \in \mathfrak{g} \otimes \Omega(M)$.
- α_λ is a Laurent polynomial in λ,

$$\alpha_\lambda = \sum_{i=a}^{b} a_i \lambda^i, \qquad a_i \in \mathfrak{g} \otimes \Omega(M).$$

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The Loop Group Interpretation

- $\Lambda G := \{\gamma : S^1 \to G\}$, loop group.
- The family F_λ can be thought of as a map either:
  - $M \times \mathbb{C}^* \to G$
  - $M \times S^1 \to G$ (for values of $\lambda \in S^1$)
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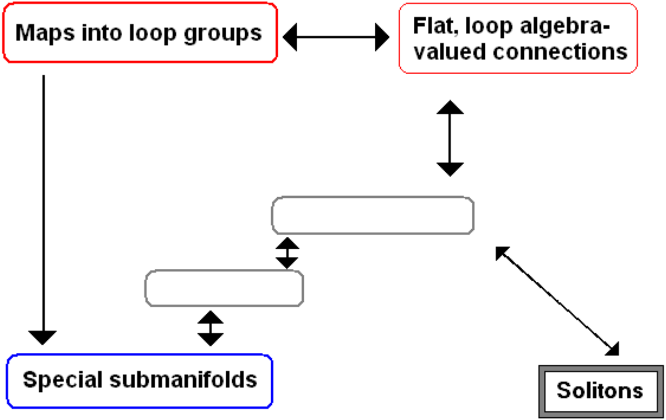
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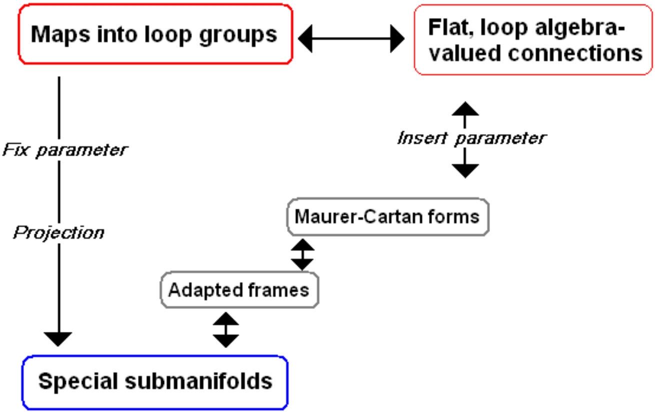
Solitons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Adapted frames
Special submanifolds

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The loop group formulation

- Any CMC surface in $\mathbb{E}^3$ admits a **conformal** parameterization:
- $f : \Sigma \to \mathbb{E}^3$, where Σ a Riemann surface
- define a function $u : \Sigma \to \mathbb{R}$
- Metric
$$ds^2 = 4e^{2u}(dx^2 + dy^2).$$
- Hopf differential Qdz^2, where
$$Q := \langle N, f_{zz} \rangle.$$
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The $SU(2)$ frame

Identify $\mathbb{E}^3$ with $\mathfrak{su}(2)$ via:

$$e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

Frame $F : \Sigma \to SU(2)$ by:

$$Fe_1 F^{-1} = \frac{f_x}{|f_x|}, \qquad Fe_2 F^{-1} = \frac{f_y}{|f_y|}.$$

The Maurer-Cartan form, α, for the frame F is defined by

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The Mauer-Cartan form

Lemma

The connection coefficients $U := F^{-1}F_z$ and $V := F^{-1}F_{\bar{z}}$ are given by

$$U = \frac{1}{2}\begin{pmatrix} u_z & -2He^u \\ Qe^{-u} & -u_z \end{pmatrix}, \qquad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u} \\ 2He^u & u_{\bar{z}} \end{pmatrix}.$$

Under the assumption H is constant, this admits an integrable *deformation*, for $\lambda \in \mathbb{S}^1$:

$$U^\lambda = \frac{1}{2}\begin{pmatrix} u_z & -2He^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix}, \quad V^\lambda = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u}\lambda \\ 2He^u\lambda & u_{\bar{z}} \end{pmatrix}.$$

The loop group frame

The family α_λ corresponds to an $\mathbb{S}^1$-family F_λ of frames for CMC surfaces. Surface corresponding to each $\lambda \in \mathbb{S}^1$ is given by the **Sym-Bobenko** formula:

$$\hat{f}^\lambda = -\frac{1}{2H}\left(Fi\sigma_3 F^{-1} + 2i\lambda\partial_\lambda F \cdot F^{-1}\right) \ .$$

The DPW method

- F_λ is a map $\Sigma \to \Lambda SU(2)$, group of loops in $SU(2)$.
- A frame for a map $\check{F} : \Sigma \to \Omega SU(2) = \Lambda SU(2)/SU(2)$
- The harmonic Gauss map, and the surface, are determined by $\check{F}$.
- **Key Point:**
  $\Omega SU(2)$ admits a complex structure
  and

$$\check{F}\big|_\lambda : \Sigma \to \mathbb{S}^2 \text{ is harmonic}$$
$$\Leftrightarrow$$
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- F_λ is a map $\Sigma \to \Lambda SU(2)$, group of loops in $SU(2)$.
- A frame for a map $\check{F} : \Sigma \to \Omega SU(2) = \Lambda SU(2)/SU(2)$
- The harmonic Gauss map, and the surface, are determined by $\check{F}$.
- **Key Point:**
  $\Omega SU(2)$ admits a complex structure
  and

$$\check{F}\big|_\lambda : \Sigma \to \mathbb{S}^2 \text{ is harmonic}$$
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The DPW method in practice

Set

$$\Lambda^+ G^{\mathbb{C}} = \{\text{loops which extend holomorphically to the unit disc } \mathbb{D}\},$$
$$\Lambda^- G^{\mathbb{C}} = \{\text{loops extending holomorphically to } \hat{C} \setminus \bar{\mathbb{D}}\}.$$

We need two loop group decompositions:

1. *Birkhoff decomposition:*

$$\Lambda^- G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}} \subset \Lambda G^{\mathbb{C}}$$

   is **open and dense** in the identity component of $\Lambda G^{\mathbb{C}}$.
2. *Iwasawa decompsition:*

$$\Lambda G^{\mathbb{C}} = \Omega G \cdot \Lambda^+ G^{\mathbb{C}}$$

   where ΩG consists of the subgroup of based loops in the real group G.

The DPW method in practice

- Given $F_\lambda : \Sigma \to \Lambda SU(2)$, extended frame for CMC surface.
- Pointwise at $z \in \Sigma$, Birkhoff decompose:

$$F_\lambda = F_- F_+, \qquad F_\pm \in \Lambda^\pm SL(2,\mathbb{C}),$$

normalization: $F_-(\lambda = \infty) = I$.

- Then F_- is a holomorphic frame for $\check{F} : \Sigma \to \Omega SU(2) \cong \Lambda SL(2,\mathbb{C})/\Lambda^+ SL(2,\mathbb{C})$.
- F_- is determined by the Maurer-Cartan form

$$\xi = F_-^{-1} \mathrm{d}F_- = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z,$$

b and $c : \Sigma \to \mathbb{C}$ holomorphic functions ("Weierstrass data").

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The DPW method in practice

- Conversely, any pair of holomorphic functions $b, c : \Sigma \to \mathbb{C}$ determines a CMC surface (The "Weierstrass representation")
- More generally, given

$$\xi = \sum_{-1}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \lambda^i dz,$$

all functions holomorphic, plus a "twisting condition",
- integrate $\Phi^{-1} d\Phi = \xi$, with $\Phi(z_0) = I$,
- Iwasawa decompose pointwise:

$$\Phi = F_\lambda G_+, \qquad F_\lambda \in \Lambda SU(2),$$

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- The holomorphic 1-form

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  called a *potential*
- Many different potentials are possible for a given surface
- **Strategy:** Seek a potential which is appropriate for a given problem.

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Solving the Björling problem

Key Point:

$$\check{F}\big|_\lambda : \Sigma \to \mathbb{S}^2 \text{ is harmonic}$$
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$\Rightarrow$

PROBLEM: Given the Björling data along a curve (f and its tangent plane), can we construct the loop group frame F_λ just along this curve?

- If so, we can (it turns out) holomorphically extend to get a holomorphic frame Φ for $\check{F}$.
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To construct F_λ, we need:

$$U^\lambda = \frac{1}{2}\begin{pmatrix} u_z & -2He^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix}, \qquad V^\lambda = \frac{1}{2}\begin{pmatrix} -u_{\bar z} & -\bar{Q}e^{-u}\lambda \\ 2He^u\lambda & u_{\bar z} \end{pmatrix}.$$

i.e., we need u, u_z and Q.

Solution

[D.B. and J. Dorfmeister: "The Björling problem for non-minimal constant mean curvature surfaces", *Comm. Anal. Geom.*, 18 (2010) 171-194]

Data: $I = (\alpha, \beta) \subset \mathbb{R}$;
$f_0 : I \to \mathbb{E}^3$;
V a vector field along I, with $\langle V, \frac{\partial f_0}{\partial x} \rangle = 0$.

Theorem

There exists a unique CMC surface which contains the curve f_0, and is tangent along this curve to the plane spanned by $\frac{\partial f_0}{\partial x}$ and V.

The holomorphic data for the loop group frame for this surface are given, on a domain in $\mathbb{C}$ containing the set $\{0\} \times I$, by the simple formulae below.

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$$u = \ln \frac{1}{2}\sqrt{\det(\frac{\partial f_0}{\partial x})}), \tag{5}$$

$$u_z = -i\Big(a + \frac{1}{2}u_x\Big), \tag{6}$$

$$Q = 2e^u\,(i\bar{b} + He^u), \tag{7}$$

Here a and b are determined from the initial data as follows: Along J we can construct an $SU(2)$ frame F from the given data (the family of tangent planes). Differentiate this along I to get the expression:

$$\hat{F}^{-1}\hat{F}_x = \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix}.$$

Remarks

- The holomorphic data can be written down explicitly
- Some geometric information of the surface can be deduced from this data
- Images of the surface can be computed numerically (software CMClab)
- Knowledge of the potential for a specific type of surface allows one to prove the existence of examples of CMC surfaces with specific properties.

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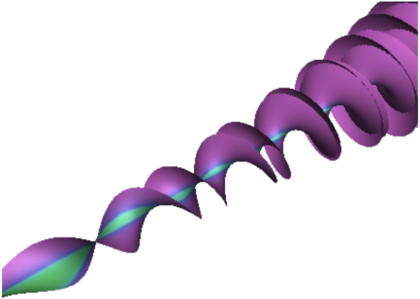
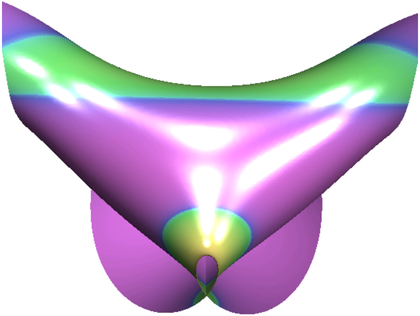
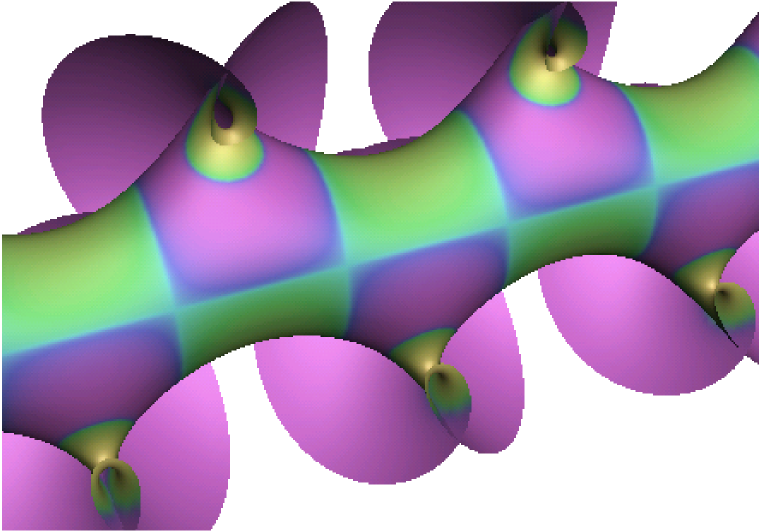
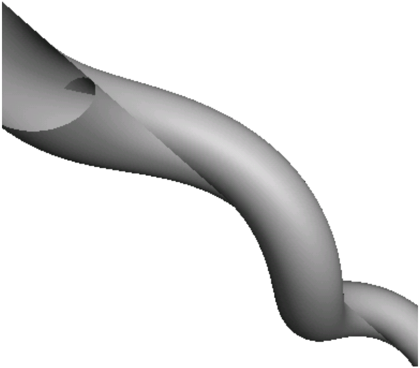
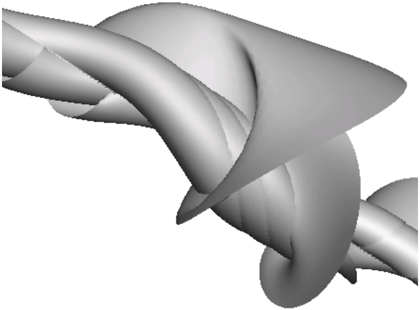
Applications: CMC surfaces which contain a straight line

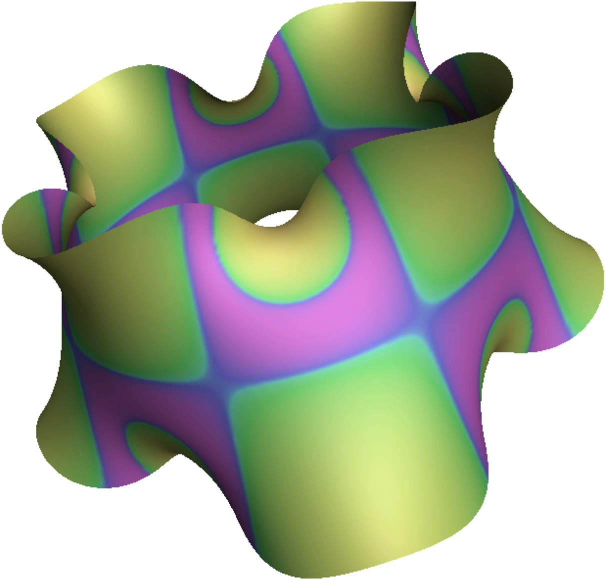


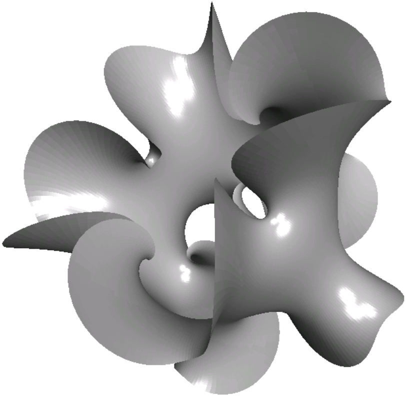


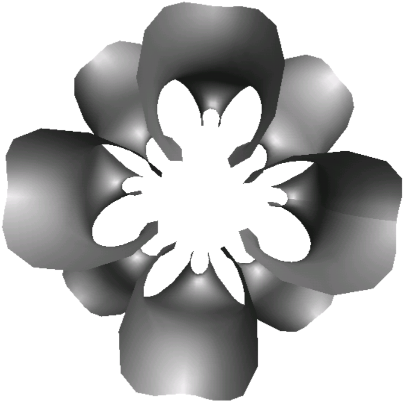


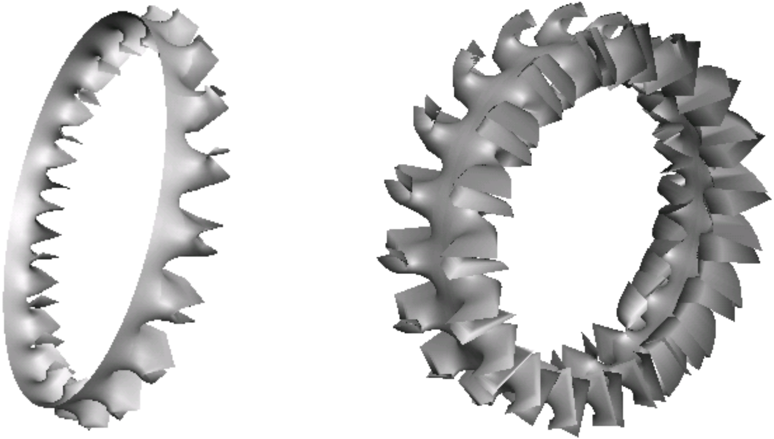



CMC surfaces which contain a circle









Outline

Other related work

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Lorentzian harmonic maps

D.B. and Martin Svensson "The Geometric Cauchy Problem for Surfaces With Lorentzian Harmonic Gauss maps" arXiv:1009.5661

- Applications: e.g. constant Gauss curvature surfaces, timelike CMC surfaces in $\mathbb{R}^{2,1}$.
- Loop group construction different: the frame F_λ is constructed froma a *pair* of potentials ξ_- and ξ_+, each a function of *one* variable only.
- Uses Birkhoff, not Iwasawa decomposition.
- *The geometric Cauchy problem can be solved for this case too* .
- Do not need real analytic initial data here.

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The geometric Cauchy problem for timelike CMC surfaces in $\mathbb{R}2, 1$

- Easy to find the potentials for surfaces of revolution.

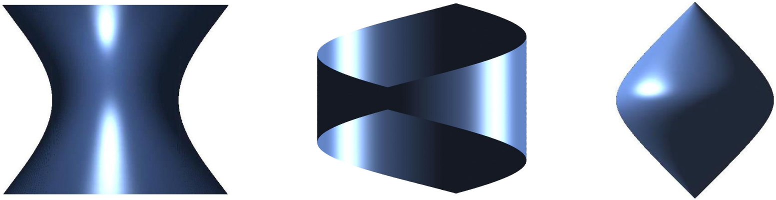

Figure: Computed from the geometric Cauchy data on a circle of radius ρ. Left: $\rho H = -1$. Center $\rho H = -1/2$. Right: $\rho H = 1$.

Pseudospherical surfaces in $\mathbb{R}^3$

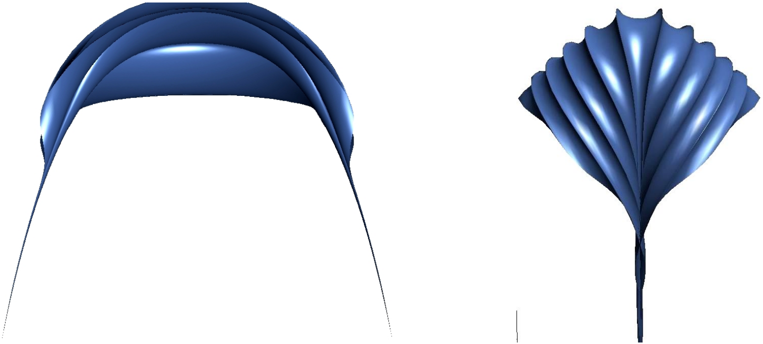

Figure: The unique K-surface containing the catenary $y = \cosh(x)$ as a geodesic principle curve

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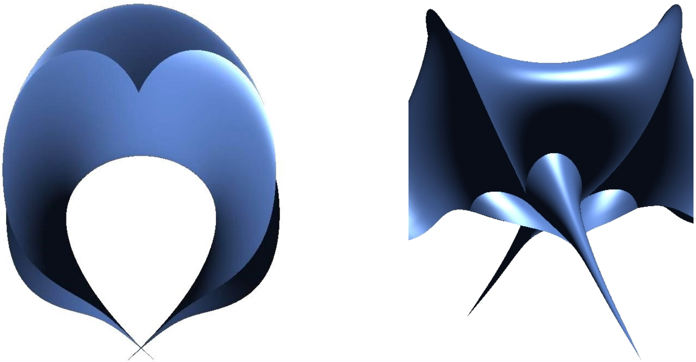

Figure: The unique K-surface containing the cubic $y = x^2(x+1)$ as a geodesic principle curve

Pseudospherical surfaces in $\mathbb{R}^3$

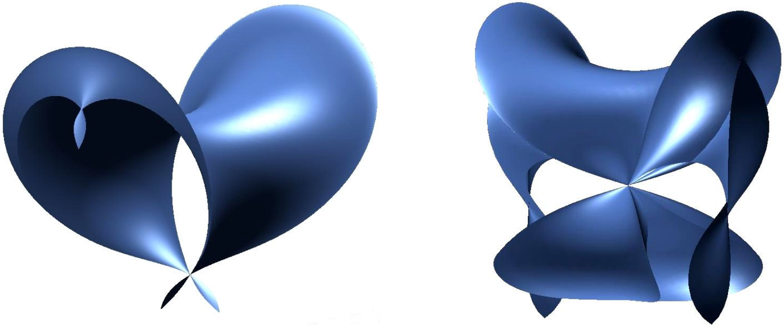

Figure: The unique K-surface containing the Bernoulli's lemniscate $(x^2 + y^2)^2 = x^2 - y^2$ as a geodesic principle curve