Manifolds and Surfaces with Locally Euclidean Metrics[1]

I.Kh. Sabitov

(Moscow State Lomonosov University)

e-mail: isabitov@mail.ru

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The metric of a Riemannian manifold M^n is called *locally Euclidean* (l.E.), if for any point $p \in M^n$ there exists a neighbourhood $U(p) \subset M^n$ which is isometric to a ball in the Euclidean space E^n with the standart metric.

Concerning l.E. metrics and surfaces with l.E. metrics one can put a lot of questions among which we would like to single the following ones.

1) Let

$$ds^2 = g_{ij} du^i \, du^j, 1 \leq i, j \leq n$$

be a given l.E. metric. How to find that local isometry between ds^2 and a ball in E^n the existence of which is guaranteed by the definition itself of l.E. metric? What can we say about the smoothness of such an isometry?

2) How to test is a given metric

$$ds^2 = g_{ij} du^i \, du^j, 1 \le i, j \le n,$$

locally Euclidean or not?

3) What are criteria for the global existence of an isometric immersion or embedding of a given n-dimensional l.E. metric in E^n?

4) What can we say about the existence/non-existence of isometric immersions and embeddings of n-dimenaional l.E. metrics in E^N with $N > n$?

5) What is known about the structure of surfaces with l.E. metrics?

6) Bendings and infinitesimally bendings of surfaces with l.E. metrics.

7) And many others.

To imagine the richness of the set of l.E. metrics and surfaces with l.E. metrics it is enough to remind the following well-known facts

1) The metric of any polyhedral surface with removed vertices is a l.E. metric. Moreover accordingly Gluck, Krigelman, Singer (1974) on any 2-dimensional piecewise linear manifold M **one can introduce a polyhedral metric with a priori known values of curvature in prescribed points including those on the boundary under the condition of satisfaction of Gauss-Bonnet equality**

$$\sum k_i + \sum e_j = 2\pi\chi,$$

where $k_i < 2\pi$ **are prescribed curvatures in some prescribed points** $p_1, \ldots, p_r, r \geq 0$, **in the interior of** M **and** $e_j < \pi$ **are prescribed exterior angles in some prescibed points** $q_1, \ldots q_s, s \geq 0$ **on the boundary of** M**. So this metric out of vertices is l.E. one.**

2) On any minimal surface S with strongly negative curvature K one can introduce a l.E. metric $\mathbf{ds^2 = \sqrt{-K} ds^2_{min}}$, where $\mathbf{ds^2_{min}}$ is the metric of the considered minimal surface (it is Ricci's criterial characteristic for the metric of a minimal surface). So any minimal surface is a support set for a l.E. metric. If on S there exist points with $K = 0$ then the above mentioned metric ds^2 will be a polyhedral one. As far as we know one can say that both of these metrics, l.E. and polyhedral, originated on minimal surfaces quiet are not studied.

3) (Rogen P., 2001) (a) In R^3 any compact surface with a non-empty boundary is isotopic to some surface with a flat metric. (b) Any two compact surfaces with flat metrics are isotopic in the class of the same type surfaces iff they are isotopic in the class of general surfaces.

1) On the smoothness of isometries.

For the material below we need essentially of the following

Theorem 1 (Calabi&Hartman+Reshetnyak+S.). ***Let two isometric*** **n*-dimensional Riemannian spaces*** **M** ***and*** **N** ***have, respectively, the metrics***

$$\mathbf{ds^2 = g_{ij}(u)du^i\ du^j \text{ and } d\sigma^2 = h_{ij}(v)dv^idv^j,}$$

of smoothness $\mathbf{C^{k,\alpha}, 0 \le k \le \infty,\ 0 \le \alpha \le 1}$***, with*** $\mathbf{k + \alpha > 0}$***. Then, any isometry*** **f** ***between them has the smoothness at least of class*** $\mathbf{C^{k+1,\alpha}}$***. If isometric Riemannian spaces are analytical, the isometry between them is also analytical.***

Accordingly this theorem we have that the smoothness of an isometry between a given l.E. metric ds^2 and the standart Euclidean metric is determined completely by the smoothness of the metric ds^2 only because the second Riemannian space, the standart Euclidean space R^n, is analytic.

2) A criterion of local Euclidity of two-dimensional metrics.

Let $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ be a C^1-smooth two-dimensional metric given in a domain D. We want to find a map $f : D \to R^2$ with

$$x = x(u, v), \;\; y = y(u, v),$$

such that $ds^2 = dx^2 + dy^2$. If such a map exists then by the previous theorem 1 it has to be of smoothness C^2. We can put

$$x_u = \sqrt{E}\cos(\varphi + \theta), \;\; x_v = \sqrt{G}\sin(\varphi - \theta), \quad (1)$$

$$y_u = \sqrt{E}\sin(\varphi + \theta), \;\; y_v = \sqrt{G}\cos(\varphi - \theta), \quad (2).$$

where

$$\cos\varphi = \frac{\sqrt{\delta + \Delta}}{\sqrt{2\delta}}, \quad \sin\varphi = \frac{F}{\sqrt{2\delta(\delta + \Delta)}},$$

$$\delta = \sqrt{EG}, \;\; \Delta = \sqrt{EG - F^2}.$$

The condition of compatibility of equations (1) and (2) leads to equations

$$\theta_u = \varphi_u + \frac{F}{2\Delta}(\ln G)_u - \frac{E_v}{2\Delta}, \;\; \theta_v = -\varphi_v - \frac{F}{2\Delta}(\ln E)_v + \frac{G_u}{2\Delta}.$$

If $\mathbf{E}, \mathbf{F}, \mathbf{G} \in \mathbf{C}^2$ the condition of compatibility of equations for $\boldsymbol{\theta}_\mathrm{u}$ and $\boldsymbol{\theta}_\mathrm{v}$ gives the equation $\boldsymbol{K} = \mathbf{0}$ which is just a classical condition of local Euclidity of a metric. In our case this condition of compatibility can be written in the form of an integral equation.

Theorem 2 (Darboux+Hartman&Wintner+S.) *A* C^1*-smooth metric*

$$\mathbf{ds}^2 = \mathbf{Edu}^2 + \mathbf{2Fdudv} + \mathbf{Gdv}^2 \qquad (*)$$

given in a circle $\boldsymbol{\Omega} : \mathbf{u}^2 + \mathbf{v}^2 \leq \mathrm{R}^2$ *is l.E. metric iff*

$$\mathbf{Im}\left(\iint_{\Omega} \frac{\mathbf{R}^2 - |\mathbf{w}|^2}{(\mathbf{z} - \mathbf{w})(\mathbf{R}^2 - \bar{\mathbf{w}}\mathbf{z})} \mathbf{H}_0(\mathbf{x}, \mathbf{y})\mathbf{dxdy}\right) = \mathbf{0},$$
$$\mathbf{z} = \mathbf{x} + \mathbf{iy} \in \boldsymbol{\Omega}, \mathbf{w} = \mathbf{u} + \mathbf{iv} \in \boldsymbol{\Omega},$$

where $\mathrm{H}_0(\mathrm{x}, \mathrm{y})$ *is an elementary function expressed explicitly through coefficients* $\mathrm{E}, \mathrm{F}, \mathrm{G}$ *and their first derivatives. Under the satisfaction of this condition the isometry* f *from* $\boldsymbol{\Omega}$ *to* R^2 *can be presented in the form of a double integral too.*

Open questions. **If a considered metric is not smooth, that is its coefficients are from classes $C^{0,\alpha}$, we don't know how to verify its local Euclidity. Further, even if we know that the metric (*) is l.E. but its coefficients are not of class C^1 then we don't know how to find an isometry to the standart R^2 using *only a finite number* of some elementary operations over E, F, G and operations of integration. This can be done only if the metric has so called isothermic form when $E = G, F = 0$. It would be interesting and usefull for applications to find such elementary criteria of local Euclidity for some special form of metrics, namely for metrics in orthogonal coordinates (when $F = 0$), in Chebyshev coordinates (when $E = G = 1, F = \cos\omega$) and in geodesic polar coordinates (when $E = 1, F = 0$).**

In multidimensional case for metrics of smoothness C^1 also there is a criterion of local Euclidity (given by A. Wintner) but it doesn't have a such simple and compact form as in the case of dimension 2. There is no also a simple expression for the existing local isometry to $R^n, n > 2$.

3) Metrics in isothermic coordinates.

The situation is much simpler if the considered metric is given in isothermic coordinates (ξ, η):

$$ds^2 = \Lambda^2(\xi, \eta)(d\xi^2 + d\eta^2). \qquad (3)$$

For this form of metrics there exists a result by Yu. Reshetnyak who proved (under very weak a priori conditions to the coefficient Λ, f.e. it is largely enough to know its continuity only) that the metric (3) is l.E. iff the function $\ln \Lambda$ is harmonic.

Then the isometry $(\xi, \eta) \mapsto (x, y)$ transfering ds^2 to $dx^2 + dy^2$ is given by a holomorphic function $z = \Phi(\zeta), \zeta = \xi + i\eta, z = x + iy$ with the condition $|\Phi'(\zeta)| = \Lambda(\zeta)$. If the domain of definition of the metric ds^2 is a circle $\Omega : |\zeta| \leq R$, then isometric immersion $z = \Phi(\zeta)$ of l.E. metric ds^2 in R^2 is given by formulae

$$\Phi'(\zeta) = \exp\left(\frac{R^2}{\pi}\iint\limits_{\Omega}\frac{\ln\Lambda(w)}{(R^2-\zeta\bar{w})^2}dudv\right),$$

$$\Phi(\zeta) = -\frac{1}{\pi}\iint\limits_{\Omega}\frac{\Phi'(w)}{\bar{w}-\bar{\zeta}}dudv,\ \ w = u + iv \in \Omega.$$

The function $\Phi(\zeta)$ is one-valued and therefore *for a l.E. metric, given in a simple-connected domain there exists always an isometric immersion in* E^2*.* As to the existence of an isometric *embedding* in E^2 it is determined by the *univalency* of the function $\Phi(\zeta)$ If the image of boundary $\partial\Omega$ is a simple curve then the map $z = \Phi(\zeta)$ gives an isometric embedding of the circle Ω in E^2 which is in the same time a conform map of Ω to the image $\Phi(\Omega)$ and the geodesic curvature k_g of the boundary $\partial\Omega$ in the metric ds^2 will be the ordinary curvature $k(s)$ of the boundary $\partial\Phi(\Omega)$. This observation leads to a problem the solution of which is equivalent to founding of a conform map of the circle to a domain with the known length of its boundary and the known curvature $k(s)$ of the boundary. Namely,

to find in the circle Ω a l.E. metric in isothermic coordinates for which the geodesic curvature $k_g(s)$ of the boundary in function of its length coincides with a given function $k(s)$ or, shortly, *to find in the circle a l.E. metric by its geodesic curvature of the boundary.* Since

$$k_g(t)=\frac{1}{\Lambda(t)}\left(1+2\mathrm{Re}(t\frac{\partial \ln \Lambda(t)}{\partial \zeta})\right), t = e^{i\varphi} \in \partial\Omega, \quad (4)$$

we have to find a harmonic function $u = \ln \Lambda(\zeta)$ by the boundary condition (4) where the left side function is a given function $k_g(t), t \in \gamma = \partial\Omega$. Let $L = 2\pi$ be the length of the boundary of a domain D to which we want to map conformally the circle Ω, and let $t = t(s)$, $0 \leq s \leq L = 2\pi$, be a map of the boundary of D in function on s to the boundary of the circle Ω in function on t. By a series of transformations we obtain that there exists a harmonic function $u = \ln \Lambda$ in Ω defined by the map $t = t(s) = e^{i\varphi(s)}$ if this map satisfies to the equation

$$\varphi(\mathrm{s}) = \frac{1}{2\pi}\int_0^{\mathrm{s}} d\mathrm{s}_1 \int_0^{2\pi} \exp\left(\mathrm{Re}\frac{1}{\pi}\int_0^{2\pi} \mathrm{k}(\sigma)\ln\frac{e^{i\varphi(\sigma)} - e^{i\varphi(\mathrm{s}_1)}}{e^{i\varphi(\sigma)} - e^{i\psi}} d\sigma\right) d\psi.$$

This equation should be solved with the condition $\varphi(2\pi) = 2\pi$. If we succeed to find a such $\varphi(s)$ then we can find $\Lambda(\zeta)$ by some explicit formules and then the map $\Phi(\zeta)$ of the circle Ω. Iff finally this map will turn be univalent (that is a schlicht function) it will be just a conform map of the circle Ω to the domain D with a given boundary. Simultaneously we obtained an algorithm for the solution of a longstanded problem asking when the natural equation of curve gives us a simple (Jordan's) curve: *so is if the holomorphe function $\Phi(\zeta)$ defined by a solution $\varphi(\mathrm{s})$ of the previous equation is univalent.*

4) The case of a multiconnected domain.

Let Ω be an (n+1)-connected domain bounded by circles $\Gamma_0 : |\zeta| = R_0, \Gamma_j : |\zeta - \zeta_k| = R_k, 1 \leq k \leq n, \zeta_1 = 0$. Then for a l.E. metric $ds^2 = \Lambda^2(\zeta)(d\xi^2 + d\eta^2), \zeta = \xi + i\eta \in \Omega$, a map $z = \Phi(\zeta)$ transfering ds^2 in $dx^2 + dy^2$ has the derivative

$$\Phi'(\zeta) = b \prod_{j=1}^{n} (\zeta - \zeta_j)^{c_{-1}^{(j)}}$$

$$\exp\left(\overline{\Phi_0(\zeta)} - \frac{1}{\pi} \iint_{\Omega} \frac{2\frac{\partial \ln \Lambda(w)}{\partial w} - \sum_j \frac{c_{-1}^{(j)}}{w - \zeta_j}}{\bar{w} - \bar{\zeta}} \, du dv \right),$$

where:

$\star$ $\Phi_0(\zeta)$ is a one-valued holomorphe function in Ω which can be presented in an explicit form using some information about the holomorphe function $\dfrac{\partial \ln \Lambda}{\partial \zeta}$ and the domain Ω;

$\star$ b is a constant;

$\star$ $c_{-1}^{(j)} = \dfrac{1}{\pi i} \oint\limits_{\Gamma_j^+} \dfrac{\partial \ln \Lambda}{\partial t} dt = -1 + \dfrac{1}{2\pi} \oint\limits_{\Gamma_j} k_g ds.$

Theorem 3 (S.) ***If all numbers*** $c_{-1}^{(j)}$ ***are integer then*** $\Phi'(\zeta)$ ***is a one-valued holomorphe function and if all Loran coefficients*** $d_{-1}^{(j)}, 1 \leq j \leq n,$ ***of*** $\Phi'(\zeta)$ ***along*** Γ_j ***are equal zero then the metric*** ds^2 ***in*** Ω ***is isometrically immersible in*** E^2 ***and the immersion is given by the holomorphe map*** $z = \Phi(\zeta)$***. If all*** $c_{-1}^{(j)}$ ***are integer but even if one of coefficients*** $d_{-1}^{(j)}$ ***is not equal zero then the metric is not immersible in*** E^2***. The same is true when even if one of numbers*** $c_{-1}^{(j)}$ ***is not integer. The univalency of the function*** $\Phi(\zeta)$ ***is necessary and sufficient for the metric*** ds^2 ***to be embeddable in*** E^2***.***

These two cases of non-immersibility of ds^2 have different geometrical nature. In the first case (all $c_{-1}^{(j)}$ are integer but there is a non-zero $d_{-1}^{(j)}$) the metric has *cylindrical type*; in the second case the metric has *conical type.* There are cases of mixed type too when near one of circles Γ_j the metric is of cylindrical type, near of others circles it is of conical type and near of third circles the metric is immersible in E^2.

It is known that in E^3 there are three types of developable surfaces so we see that from the point of view of the interior geometry there are only two types of l.E. metrics.

An example.

For the metric $ds^2 = \rho^b(d\xi^2 + d\eta^2), \rho^2 = \xi^2 + \eta^2$ given in a ring $\Omega : R_1 \leq |\zeta| \leq R_2$ we have

$$\Phi'(\zeta) = \zeta^b, \Phi(\zeta) = \frac{\zeta^{b+1}}{b+1}, b \neq -1; \Phi(\zeta) = \ln\zeta, b = -1.$$

If $b \neq -1$ is an integer then c_{-1} is an integer, $d_{-1}=0$, and the metric is immersed in E^2 as a $|b+1|$ times covered ring; if $b = -1$, then c_{-1} is an integer but $d_{-1} = 1 \neq 0$, and the metric is not immersible in E^2 (the image of Ω is a rectangle which corresponds to a cylindr in E^3); if b is not integer then the metric is not immersible in E^2 too (the image of Ω is circular sector which corresponds to a cone in E^3).

5) Surfaces in E^3 with l.E. metrics.

A surface in E^3 is called *developable* if its metric is l.E. In the classical case a developable surface is ruling one with tangent planes stationary along the generatrices. But it is true under some additional conditions of smoothness only. The first example of a developable surface not containing straightline generatrices was given by Lebesgue. His surface has a tangent plane in any point but it is not C^1-smooth. 50 years after Nash and Kuiper proved the existence of such C^1-smooth surfaces. It turns out that even an additional condition of $C^{1,\alpha}$-smoothness doesn't change the situation: Yu.F. Borisov showed that the Euclidean plane admits isometric bendings in the class of $C^{1,\alpha}$-smooth surfaces with $\alpha < 1/7$ such that the obtained surfaces don't contain any straightline segment. But if the exponent α is sufficiently large then the situation changes: Borisov proved that for $\alpha > 2/3$ any $C^{1,\alpha}$-smooth surface with a l.E. metric has the ruling structure.

Meanwhile there exist other characteristics of C^1-smooth surfaces with flat metrics which guarantee their ruling structure too. Namely for C^1-smooth surfaces two notions of exterior curvature are known, one is due to A.V. Pogorelov and it gives so called *surfaces of bounded exterior curvature in Pogorelov sense* (BECP), the second notion was introduced by Yu.D. Burago and it gives *surfaces of bounded exterior positive curvature in Burago sense* (BEPCB). Let's introduce now the notion of a *torse type* surface: it is a C^1-smooth surface such that through any its point at least one rectilinear segment passes along which the tangent plane to the surface is stationar that is it is the same along a generatrix. So two results are known. I present them in the form of a table

Pogorelov	Burago&Shefel'
(l.E.+BECP) $\Rightarrow$ torse	(l.E.+BEPCB) $\Rightarrow$ torse
(torse+BECP) $\Rightarrow$ l.E.	torse $\Rightarrow$ (l.E.+BEPCB)

We can give a more clear description of torse type surfaces with l.E. metrics. A point on a torse-type surface S is called *planar* if it has a plane neighbourhood on S. Further we consider torse type surfaces without planar points only. For them one can prove that by any point only one generatrix passes. Then on such surfaces we can introduce at least "in small" so called *asymptotic parametrization* for which there is the following

Theorem 4 (Hartman&Nirenberg). ***If a developable surface*** $\mathbf{S} \in \mathbf{C^n}, \mathbf{2} \leq \mathbf{n} \leq \infty$***, doesn't contain any plane domain then in a neighbourhood of any its point the position vector of the surface can be presented in the form***

$$\mathbf{S} : \mathbf{R}(\mathbf{u}, \mathbf{t}) = \mathbf{r}(\mathbf{u}) + \mathbf{t}\mathbf{l}(\mathbf{u}), \qquad (5)$$

where $\mathbf{r} = \mathbf{r}(\mathbf{u}) \in \mathbf{C^n}$ ***is position vector of a directix,*** $\mathbf{l}(\mathbf{u}) \in \mathbf{C^0}$ ***is direction vector of generatrices and in the general case*** $\mathbf{l}(\mathbf{u}) \notin \mathbf{C^1}$***.***

The situation, when $\mathbf{l(u)} \in \mathbf{C}$ but $\mathbf{l(u)} \notin \mathbf{C^1}$, can be really even for surfaces $\mathbf{S} \in \mathbf{C^\infty}$; the example of such a surface is given in the Klingenberg's textbook on Differential Geometry. Hartman and Nirenberg had affirmed that such a situation can occur even in analytic case but V. Ushakov have found a gap in their argumentation and he have proved that *on any analytical surface with l.E. metric one can introduce an analytical asymptotic parametrization.*

We know many geometrical properties of $\mathbf{C^1}$-smooth torse type surfaces and some geometrical conditions for $\mathbf{C^1}$-smooth surfaces to be torse type surfaces but due to the possibilty for $\mathbf{l(u)}$ not to be in $\mathbf{C^1}$-class we can't verify whether a ruling surface given in the very natural and simple form (5) is a torse type surface because we can find neither its metric form and no calculate its normal vector to check is it constant along a generatrixe or not. In other words there was no an analytic apparatus for working with torse type surfaces.

But it turns out that in reality we can say much more about the smoothness property of an asymptotic parametrisation. Namely the following theorem is true

Theorem 5 (S., 2009). ***On a*** C^1***-smooth torse type surface for any point there exists a neighbourhood where one can introduce an asymptotic parametrization (5) with*** C^1***-smooth directrix line*** $\boldsymbol{\rho}(\mathbf{u})$ ***and*** $\mathrm{C}^{0,1}$***-class field of unit vectors*** $\mathbf{l}(u)$ ***of generatrices satisfying the equation***

$$(\boldsymbol{\rho}'(\mathbf{u}), \mathbf{l}(\mathbf{u}), \mathbf{l}'(\mathbf{u})) = \mathbf{0} \qquad (\mathbf{6})$$

at all points of existence of derivative $\mathbf{l}'(\mathbf{u})$***.***

Moreover the inverse is true too:

Theorem 6 (S., 2009). ***A ruling surface having an asymptotic parametrization (5) where*** $\boldsymbol{\rho}(\mathbf{u}) \in \mathbf{C}^1, \mathbf{l}(\mathbf{u}) \in \mathbf{C}^{0,1}$ ***with the satisfaction of the equation (6) at all points of existence of*** $\mathbf{l}'(\mathbf{u})$ ***is a*** C^1***-smooth torse type surface.***

What is not trivial in this theorem this is the fact that in the presentation (5) itself the position vector $R(u, t)$ is not C^1-smooth: its derivative R'_u doesn't existe everywhere (but in the contrary the derivative R'_t is even in $C^{0,1}$-class). So C^1-smoothness of S means that there is an other parametrization, here the presentation of S in the form $z = f(x, y)$, in which the surface has C^1-smoothness.

So we have a complete analytical description of C^1-smooth torse type surfaces. And now we can give some additional geometrical properties of them.

Theorem 7 (S., 2010).*For any* C^1*-smooth torse type surface* S *there exists a* C^1*-smooth cone* C *such that the spherical images of* S *and* C *coincide.*

Corollary.*A* C^1*-smooth torse type surface is a surface of class BECP.*

***Remark.* As we observed above for l.E. metrics in multi-connected domains there are only two types – cylindrical and conic metrics. May be this phenomenon is explained by theorem 7.**

6) Isometric immersions and embeddings of l.E. metrics in E^3 and E^4.

A rather simple sufficient condition of isometric embeddebality of l.E. metrics in E^3 is given by the following

Theorem 8 (S.). *Let*

$$ds^2 = \Lambda^2(\xi, \eta)(d\xi^2 + d\eta^2) \qquad (7)$$

be a l.E. metric given in a n*-connected domain* $\bar{D}_n$ *bounded by* $n \geq 1$ *circles. Suppose that* $\Lambda \in C^{m,\alpha}(\bar{D}_n), m \geq 1, 0 < \alpha < 1$ *and that* $\bar{D}_n$ *is isometrically immersible in* E^2*. Then the domain* D_n *with the metric (7) can be isometrically embedded in* E^3 *as a developable surface of smoothness* C^∞ *in the open domain* D_n *and of smoothness* C^{m+1} *in* $\bar{D}_n$*.*

The proof of theorem is constructive and the searched surface is obtained as a cylindric or conic surface. Since any simple connected compact domain $\bar{D}_1$ is always immersible in E^2 we can affirm that *any such domain with a l.E. metric can be isometrically embedded in* E^3.

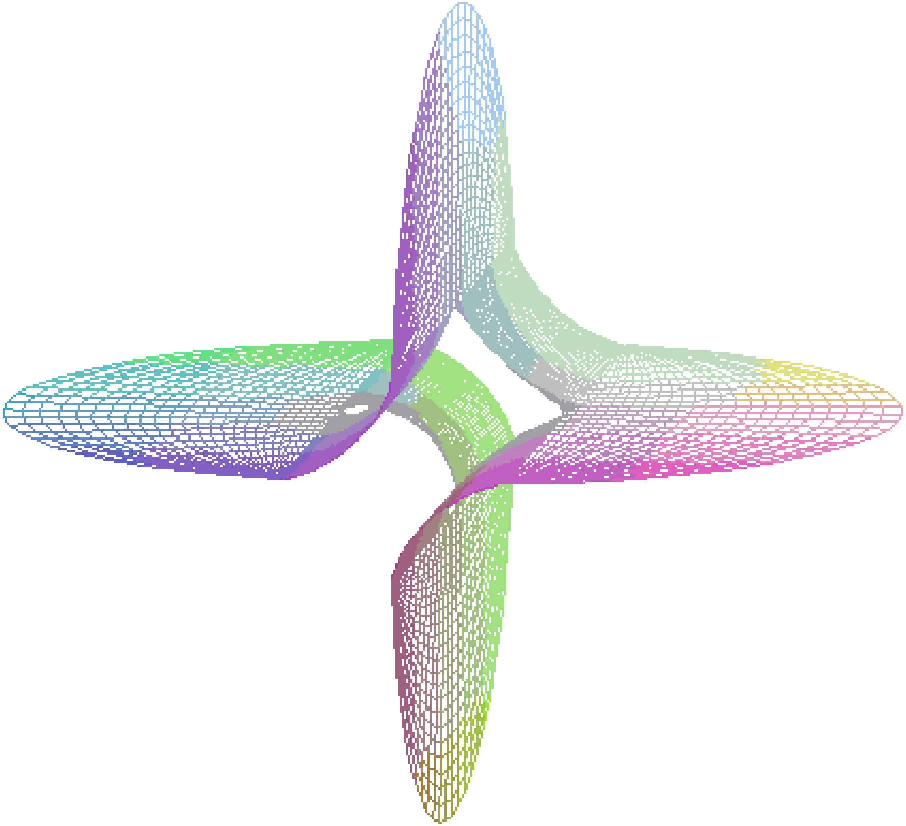
As an example let's consider the question on the isometric embedding in E^3 for the metric

$$ds^2 = 4(\xi^2 + \eta^2)(d\xi^2 + d\eta^2),$$

given in the ring $D_2 : R_1^2 \leq \xi^2 + \eta^2 \leq R_2^2$. This metric is immersible in E^2 (but not embeddable in E^2) by the map $z = \zeta^2$ as the double covered ring so it is embeddable in E^3. In general for l.E. metrics in a compact ***two-connected*** domain one can be proven that they are ***immersible*** in E^3 while the problem of their embeddibility is open .

As to l.E. metrics given in $n \geq 3$-connected domains we can present an example of a l.E. metric in 3-connected domain which is not immersible in E^3 even in the class of C^1-smooth torse type surfaces.

In general the question on sufficient conditions of not-immersiabilty of l.E. metrics in E^3 remains open as well as the question about the form of existing immersions (cylinders, cones or development of tangents to a spatial curve).



For n $\geq$ 3-dimensional l.E. metrics the question on their isometric embeddability in a space E^N is open even for l.E. metrics given in a closed n-ball B_n, in the sense that we don't know the minimal dimension $N_0(n)$ of the space E^{N_0} in which one can embed any l.E. metric given in B_n. There is no also any results concerning the "natural" presentation of n-dimensional l.E. metrics as the standart metric in corresponding domains in E^n.

Remark. **We would like to remark that seeming questions can be posed for other metrics of constant curvature. It is natural to try to "see" an abstractly given hyperbolic or spherical metric presenting it as the standart metric of a domain in the hyperbolic or spherical space.**

Now we are passing to immersions and embeddings of some *classic* l.E. metrics. Let's consider the question on the isometric realization of a *Möbius strip.* I can propose to your attention only some results and questions. At first, the study of the midline of a *standart Möbius band*, an isometric realisation of a rectangle Möbius strip in R^3, gave the following remarkable observation: *all closed analytical spatial curves can be divided in two non-intersecting classes – with periodic or antiperiodic principal normals and binormals in the Frenet trihedron of the curve*; in the second case we call the curve *semiperiodic* (because its tangent is always periodic). The midline of a standart Möbius band with a flat metric should be always semiperiodic with some additional properties of its curvature and torsion and it gives a unique distribution of the generatrices of the surface (by the way there is a very interesting intrepretation of their distribution from the point of view of the theory of elasticity).

A second remark concerns the question on the simplest algebraic equation of a Möbius surface with a flat metric. In a work by Wunderlich (1962) it is shown the existence of an algebraic surface of degree 39 a part of which presents a Möbius band. The first explicit parametric equation of a Möbius band found by G. Shcwarz (1990) gives an algebraic surface of degree 20. Finally in 2007 we have found an equation

$$\begin{aligned}
&125(x^2+y^2)^2\mathbf{z^3}+25y[4x-16(x^2+y^2)+\\
&18x(x^2+y^2)-3(x^2+y^2)^2]\mathbf{z^2}+\\
&5(1-16x+105x^2+41y^2-360x^3-\\
&292xy^2+675x^4+779x^2y^2+131y^4+\\
&270xy^4-918y^2x^3-648x^5+141x^2y^4+\\
&381y^2x^4+243x^6+3y^6)\mathbf{z}+5y+80xy+\\
&525yx^2+165y^3+1160xy^3-1800yx^3+\\
&2855x^2y^3+3375yx^4+169y^5+360xy^5-\\
&2880x^3y^3+191x^2y^5-3240yx^5+1025y^3x^4+\\
&1215yx^6-y^7=0
\end{aligned}$$

of degree 7 on which a Möbius band is situated. For the moment it is a surface of minimal degree for a Möbius band with a flat metric.

Having this equation immediately the following questions arise: 1) what is an algebraic surface of minimal degree for a standart Möbius band and what is it for a general Möbius surface with a flat metric; 2) how to describe all algebraic surfaces with a flat metric and how to distinguish among them non-orientable ones?

Now about the realizations of complete l.E. metrics. Again we'll speak only about Möbius bands. Following Blanusha, we can indicate in $\mathbf{R}^4$ a surface

$$x_1 = \rho \cos\left(\frac{u}{2} + h(\rho)\right), \ \ x_2 = \rho \sin\left(\frac{u}{2} + h(\rho)\right),$$

$$x_3 = \frac{\sqrt{4-\rho^2}}{2}\cos(u + H(\rho)), x_4 = \frac{\sqrt{4-\rho^2}}{2}\sin(u + H(\rho)),$$

where $0 \leq u \leq 2\pi$ **and** $h(\rho) \ \ H(\rho)$ – **some even analytic functions defined in an interval** $-R < \rho < R \leq 2$**. By a direct calculation one can verify that this surface is homeomorphic to the rectangle** $P : [0, 2\pi] \times (-R, R)$ **with the identification of points** $(0, \rho)$ **and** $(2\pi, -\rho)$**. Further for a special choice of** h **and** H **its metric will be flat and complete in** P**. It is easy to check that this sur-**

face is situated on the 3-dimensional ellipsoid with the equation

$$x_1^2 + x_2^2 + 4(x_3^2 + x_4^2) = 4.$$

In the same time it is known that on the sphere $\mathbf{S^3} \in \mathbf{E^4}$ one can exist none non-orientable complete surface with a l.E. metric. So we arrive to the necessity to study this phenomenon: on what kind of ellipsoids such a surface can be situated, on any ellipsoids or only on those that are suffciently far from a sphere?

Finally a few words concerning some other problems. At first one can ask about the solutions "in whole" of the "simplest" Monge-Ampére equation

$$z_{xx}z_{yy} - z_{xy}^2 = 0.$$

It is known that any C^2-solution $\mathbf{z(x, y)}$ of this equation over $\mathbf{R^2}$ is a cylinder. But if we consider solutions over R^2 with a finite number of removed points or lines then we don't know anything about their nature as well as about solutions of equations $\mathbf{Hess(z(x_1, \dots, x_n)) = 0}$ over $\mathbf{R^n, n > 2}$, with a simple set removed points.

As to bendings (continuous isometric deformations) of surfaces with l.E. metrics here the most interesting result by my opinion is the following one: recently M.I. Shtogrin showed that some Platonian polyhedra admit bendings in the class of piecewise smooth surfaces (for the moment it is done for tetrahedra, octahedra and cubes.) This means that during bending these surfaces composed by a number of smooth parts which are cylinders or cones (in the initial position they are plane faces).

And the last remark. Among many other questions I would like to mention the following one: let L be a spatial curve, how to construct a regular developable surface with the boundary L? It is an analogue of Plateau problem for minimal surfaces when we have to find a surface with a given boundary and having the mean curvature $H = 0$, in our problem we have to find a surface with a given boundary and having the Gaussian curvature $K = 0$. Evidently this problem will have many practical applications in the engineering and architecture.