Construction of complete (hyperbolic) minimal surfaces in $\mathbb{R}^3$

Antonio Alarcón

Departamento de Geometría y Topología

Universidad de Granada

Definition

- An open Riemann surface M is said to be hyperbolic if it carries non-constant negative subharmonic functions.
- Otherwise, M is said to be parabolic.

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Conjecture (Meeks-Sullivan)

Proper minimal surfaces of finite topology are parabolic.

Conjecture (Schoen-Yau 1985)

Minimal surfaces properly projecting into a plane are parabolic.

Conjecture (Calabi 1966)

A complete minimal surface has no bounded coordinate function.

- Jorge-Xavier 1980 There exists a complete minimal surface contained in a slab of $\mathbb{R}^3$.

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- Weierstrass Representation $X = (X_j)_{j=1,2,3} : M \to \mathbb{R}^3$ conformal minimal immersion

$$
\begin{aligned}
X &= \Re \int \left(\frac{1}{2}\left(\frac{1}{g} - g\right)\Phi_3 \,,\; \frac{\imath}{2}\left(\frac{1}{g} + g\right)\Phi_3 \,,\; \Phi_3 \right) \\
\Phi_3 &= \partial_z X_3 \text{ (holomorphic 1−form)} \\
g &= \text{st. proj. Gauss map (meromorphic function)} \\
ds^2 &= \frac{1}{4}\left(\frac{1}{|g|} + |g|\right)^2 |\Phi_3|^2
\end{aligned}
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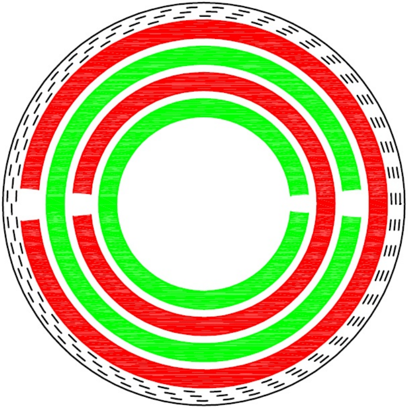

- $M = \mathbb{D}$
- $\Phi_3 = dz$
- $|g|$ bigger and bigger in the compact sets
- $ds^2 = \frac{1}{4}(\frac{1}{|g|} + |g|)^2 |dz|^2$

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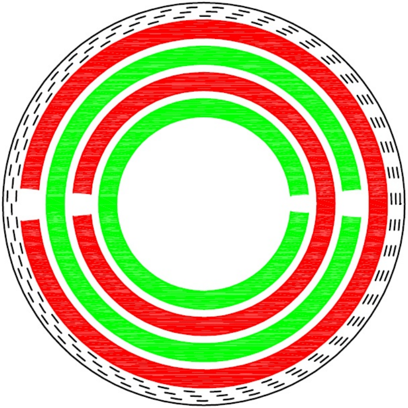

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- Runge's Theorem 1948 (1885) A holomorphic function on a Runge compact subset of an open Riemann surface M can be uniformly approximated by holomorphic functions on M.

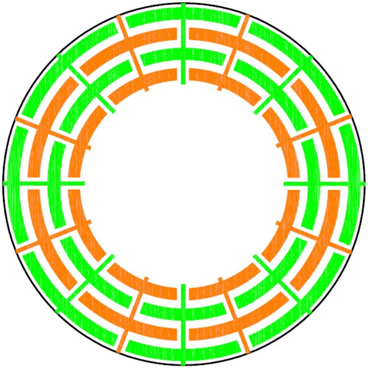
Conjecture (Calabi 1966)

A complete minimal surface can not be bounded.

Conjecture (Hadamard 1898)

A complete negatively curved surface in $\mathbb{R}^3$ can not be bounded.

- Nadirashvili 1996 There exists a complete negatively curved minimal surface contained in a ball of $\mathbb{R}^3$.



Question (Yau 2000)

- *Which topological types admits a complete bounded minimal surface?*
- *Are there complete minimal surfaces properly immersed in a ball?*

- López-Martín-Morales 2002 There are complete bounded minimal surfaces of arbitrary finite topological type. (Period Problem.)
- Morales 2003 There exists proper hyperbolic minimal disc in $\mathbb{R}^3$. (Counterexample to Meeks-Sullivan's conjecture.)
- AA, Ferrer, López, Martín, Morales Many examples.

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- Tools
  - Nadirashvili's technique for the completeness.
  - Morales' technique for the properness.
  - The Bridge Principle for minimal surfaces for the arbitrary topological type.

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  - Nadirashvili's technique for the completeness.
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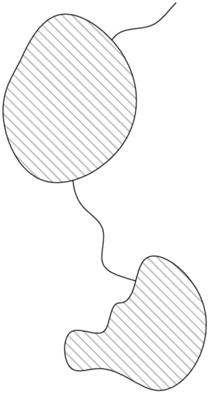
- Properness (in $\mathbb{R}^3$) does not influence the topology of immersed (hyperbolic) minimal surfaces.
- Conformal structure?

The Approximation Lemma

- Given an open Riemann surface $\mathcal{M}$, a compact subset $S \subset \mathcal{M}$ is said to be admissible iff
  - S is Runge,
  - $M_S := \overline{S^\circ}$ consists of a finite collection of pairwise disjoint compact regions in $\mathcal{M}$ with analytical boundary,
  - $C_S := \overline{S - M_S}$ consists of a finite collection of pairwise disjoint analytical Jordan arcs.



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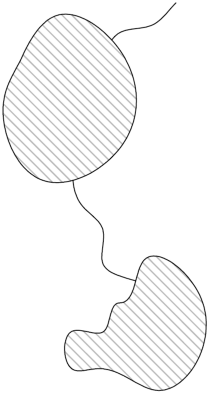
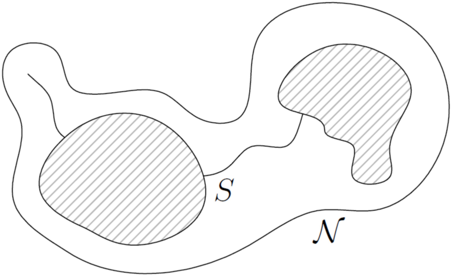
- Let $\mathcal{M}$ be an open Riemann surface and $S \subset \mathcal{M}$ be an admissible subset. A smooth map $X = (X_j)_{j=1,2,3}$ is said to be a generalized conformal minimal immersion (GCMI) iff
  - $X|_{C_S}$ is an immersion, and
  - $X|_{\mathcal{M}_S}$ is a conformal minimal immersion.



Lemma (The Approximation Lemma) (AA-López 2009)

Let $\mathcal{N}$ be an open Riemann surface of finite topology, and let S be a connected admissible compact subset in $\mathcal{N}$. Let $X = (X_1, X_2, X_3) : S \to \mathbb{R}^3$ be a GCMI on S.

Then X can be uniformly approximated on S by a sequence of conformal minimal immersions $Y(n) = (Y(n)_1, Y(n)_2, Y(n)_3) : \mathcal{N} \to \mathbb{R}^3$.



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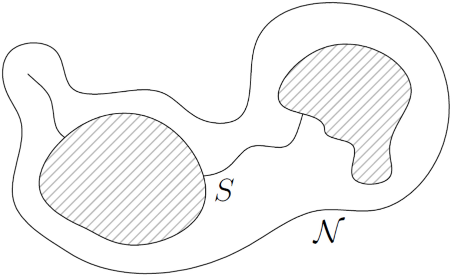
Then X can be uniformly approximated on S by a sequence of conformal minimal immersions $Y(n) = (Y(n)_1, Y(n)_2, Y(n)_3) : \mathcal{N} \to \mathbb{R}^3$.
Moreover, we can choose the third coordinate function $Y(n)_3 = X_3\ \forall n$ provided that X_3 extends harmonically to $\mathcal{N}$ and $\partial_z X_3$ never vanishes on C_S.



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- Mergelyan-Bishop's Theorem 1958 Let M be an open Riemann surface, let $K \subset M$ be a compact Runge set and let $f : K \to \mathbb{C}$ be a continuous function which is holomorphic in K°. Then f can be uniformly approximated on K by functions holomorphic in M.
- Implicit Function Theorem.

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$$X_n : M_n \to \mathbb{R}^3, \quad \|X_n - X_{n-1}\| < \epsilon_n \text{ on } M_{n-1}.$$

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- Global properties?

Completeness

Theorem (AA-Fernández-López 2009)

Let M be an open Riemann surface and let $u : M \to \mathbb{R}$ be non-constant harmonic function.
Then there exists a conformal complete minimal immersion $Y = (Y_1, Y_2, Y_3) : M \to \mathbb{R}^3$ with $Y_3 = u$.

Theorem (AA-Fernández-López 2009)

Let M be an open Riemann surface and let $u : M \to \mathbb{R}$ be non-constant harmonic function.
Then there exists a conformal complete minimal immersion $Y = (Y_1, Y_2, Y_3) : M \to \mathbb{R}^3$ with $Y_3 = u$.

Proof. Construct a sequence of conformal minimal immersions $X_n = (X_{n,1}, X_{n,2}, X_{n,3}) : M_n \to \mathbb{R}^3$ with

1. $\|X_n - X_{n-1}\| < \epsilon_n$ on M_{n-1}.
2. $X_{n,3} = u|_{M_n}$.
3. $\mathrm{dist}_{M_n}(P_0, \partial M_n) > n$. ($P_0 \in (M_1)^\circ$.)

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- Consider in $M_n^\circ - M_{n-1}$ a Jorge-Xavier's type labyrinth K_n and apply the AL to the map $X : M_{n-1} \cup K_n \to \mathbb{R}^3$ given by
  - $X|_{M_{n-1}} = X_{n-1}$
  - $X|_{K_n}$ has Weierstrass data

$$\Phi_3 = \partial_z u, \quad |g| \text{ large enough.}$$

Q.E.D.

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Q.E.D.

- Completeness does not influence the underlying conformal structure of immersed minimal surfaces (López 2009).

- Fujimoto 1988 The Gauss map of a complete non-flat minimal surface in $\mathbb{R}^3$ can not omit more than 4 points in $\mathbb{S}^2$.
- The Gauss map of a conformal complete non-flat minimal immersion $X : \mathbb{C} \to \mathbb{R}^3$ can not omit more than 2 points in $\mathbb{S}^2$. (Picard's Theorem.)

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Corollary

Let M be an open Riemann surface.
Then there exists a conformal complete non-flat minimal immersion $Y : M \to \mathbb{R}^3$ whose Gauss map omits 2 values of $\mathbb{S}^2$.

Proof. Take a harmonic function $u : M \to \mathbb{R}$ such that $\partial_z u$ never vanishes on M and apply the theorem above. **Q.E.D.**

Corollary

A necessary and sufficient condition for an open Riemann surface to admit a conformal complete minimal immersion into a slab of $\mathbb{R}^3$ is to carry non-constant bounded harmonic functions.

Proof. Take a non-constant bounded harmonic function $u : M \to \mathbb{R}$ and apply the theorem above. **Q.E.D.**

Properness

Theorem (AA-López 2009)

Let M be an open Riemann surface and let $\theta \in (0, \frac{\pi}{4})$. Then there exists a conformal minimal immersion $Y = (Y_1, Y_2, Y_3) : M \to \mathbb{R}^3$ such that

$$Y_3 + \tan(\theta)|Y_1| : M \to \mathbb{R} \text{ is positive and proper.}$$

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1. $\|X_n - X_{n-1}\| < \epsilon_n$ on M_{n-1}.
2. $X_{n,3} + \tan(\theta)|X_{n,1}| > n$ on ∂M_n.
3. $X_{n,3} + \tan(\theta)|X_{n,1}| > n - 1$ on $M_n - M_{n-1}$.

Q.E.D.

- Hoffman-Meeks 1990 The only proper minimal surfaces in $\mathbb{R}^3$ contained in a half-space are planes.

Corollary (Schoen-Yau's Conjecture)

Any open Riemann surface admits a conformal minimal immersion in $\mathbb{R}^3$ properly projecting into a plane.

Proof. Observe that $(Y_1, Y_3) : M \to \mathbb{R}^2$ is proper. **Q.E.D.**

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- Properness (in $\mathbb{R}^3$) does not influence the underlying conformal structure of immersed minimal surfaces.

- Can "plane" be changed by "convex domain in $\mathbb{R}^2$" + complete?

The Calabi-Yau Problem in $\mathbb{C}^3$

- Let $\mathcal{N}$ be an open Riemann surface. A null curve in $\mathbb{C}^3$ is a holomorphic immersion $F = (F_1, F_2, F_3) : \mathcal{N} \to \mathbb{C}^3$ such that

$$(dF_1)^2 + (dF_2)^2 + (dF_3)^2 = 0.$$

- $F : \mathcal{N} \to \mathbb{C}^3$ null curve $\Leftrightarrow \Re(F), \Im(F) : \mathcal{N} \to \mathbb{R}^3$ conformal minimal immersions. Furthermore,

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$$ds_F^2 = 2ds^2_{\Re(F)} = 2ds^2_{\Im(F)}.$$

- Calabi-Yau problem in $\mathbb{C}^3$ Are there complete bounded null curves in $\mathbb{C}^3$?
- Martín-Umehara-Yamada 2009 There exists a complete bounded null curve with the conformal type of the disk.

- $k \in \{2, 3, 4, 5, 6\}$.
- $\Omega \subset \mathbb{R}^k$ convex domain.
- $[\mathbf{\Omega}] = \Omega \times \mathbb{R}^{6-k} \subset \mathbb{C}^3 \equiv \mathbb{R}^6$.

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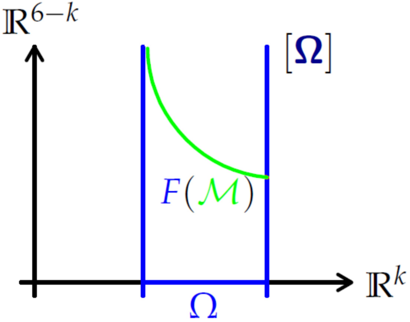
Theorem (AA-López 2010)

Let M be an open orientable surface.
Then there exists an open Riemann surface $\mathcal{M} \cong M$ and a complete null curve $F : \mathcal{M} \to [\boldsymbol{\Omega}] \subset \mathbb{C}^3$ properly projecting into Ω.



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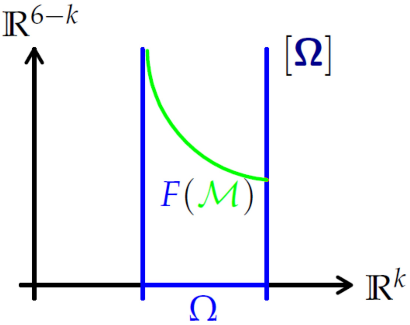

- Let M be an open orientable surface.

Corollary (Calabi-Yau Problem in $\mathbb{C}^3$)

Let Ω be a convex domain in $\mathbb{C}^3$.
Then there exists an open Riemann surface $\mathcal{M} \cong M$ and a proper complete null curve $F : \mathcal{M} \to \Omega$. Furthermore, if $\Omega = \mathbb{C}^3$ then $\mathcal{M}$ can be prescribed.

Proof. $k = 6$. **Q.E.D.**

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Proof. $k = 6$. **Q.E.D.**

Corollary

There exists a bounded complete minimal surface $X : M \to \mathbb{R}^3$ such that all its associate surfaces are well defined and bounded.

Proof. Choose Ω bounded. **Q.E.D.**

- Calabi-Yau problem in $\mathbb{H}^3$ Are there complete bounded CMC-1 surfaces in $\mathbb{H}^3$?
- Martín-Umehara-Yamada 2009 A simply connected one.

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Corollary (Calabi-Yau Problem in $\mathbb{H}^3$)

There exists a complete bounded CMC-1 immersion $X : M \to \mathbb{H}^3$.

Proof. Use a transformation in explicit coordinates due to Martín-Umehara-Yamada that applies complete null curves in $\mathbb{C}^3 - \{z_3 = 0\}$ into complete bounded CMC-1 surfaces in $\mathbb{H}^3$. **Q.E.D.**

- Bourgain 1993 There are no complete bounded null curves in $\mathbb{C}^2$.
- Calabi-Yau problem in $\mathbb{C}^2$ Are there complete bounded complex curves in $\mathbb{C}^2$?
- Jones 1979 A simply connected one.
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Corollary (Calabi-Yau Problem in $\mathbb{C}^2$)

Let Ω be a convex domain in $\mathbb{C}^2$.
Then there exists an open Riemann surface $\mathcal{M} \cong M$ and a proper complete holomorphic immersion $F : \mathcal{M} \to \Omega$.
Furthermore, if $\Omega = \mathbb{C}^2$ then $\mathcal{M}$ can be prescribed (Bishop 1961).

Proof. $k = 4$. **Q.E.D.**

Corollary (Calabi-Yau Problem in $\mathbb{R}^3$) (Ferrer-Martín-Meeks 2009)

Let Ω be a convex domain in $\mathbb{R}^3$.
Then there exists a proper complete minimal immersion $X : M \to \Omega$.

Proof. $k = 3$. **Q.E.D.**

Corollary (Calabi-Yau Problem in $\mathbb{R}^3$) (Ferrer-Martín-Meeks 2009)

Let Ω be a convex domain in $\mathbb{R}^3$.
Then there exists a proper complete minimal immersion $X : M \to \Omega$.

Proof. $k = 3$. **Q.E.D.**

Corollary (Original Aim)

Let Ω be a convex domain in $\mathbb{R}^2$.
Then there exists a complete minimal immersion $X : M \to \mathbb{R}^3$ properly projecting into Ω.

Proof. $k = 2$. **Q.E.D.**

- Let $\mathcal{M}$ be an open Riemann surface and $S \subset \mathcal{M}$ an admissible subset. A smooth map $F = (F_1, F_2, F_3) : S \to \mathbb{C}^3$ is said to be a generalized null curve iff
  - $\sum_{i=1}^{3} (dF_i)^2 = 0$,
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Lemma (The Approximation Lemma)

Let $\mathcal{N}$ be an open Riemann surface of finite topology, and let S be a connected admissible compact subset in $\mathcal{N}$. Let $F = (F_1, F_2, F_3) : S \to \mathbb{C}^3$ be a generalized null curve.
Then F can be uniformly $\mathcal{C}^1$-approximated on S by a sequence of null curves $H_n : \mathcal{N} \to \mathbb{C}^3$.
Moreover, we can choose the third coordinate $(H_n)_3 = F_3\ \forall n$ provided that F_3 extends holomorphically to $\mathcal{N}$ and dF_3 never vanishes on C_S.

To construct a complete proper null curve $F : \mathbb{D} \to \mathbb{B} \subset \mathbb{C}^3$ we apply recursively the following technical result.

Lemma

Let M be a simply connected compact region in $\mathbb{D}$ with $0 \in M^\circ$, let $\xi > 0$, $\rho > 0$, $n \in \mathbb{N}$, and $X : M \to \mathbb{C}^3$ be a null curve satisfying

$$\rho - \xi < \|X(P)\| < \rho \quad \forall P \in \partial M.$$

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2. $\|Y - X\| < \epsilon / 2^n$ *on* M,
3. $\rho - \xi < \|Y(P)\|\ \forall P \in M' - M^\circ$,
4. $\mathrm{dist}_{(M',Y)}(0, \partial M') > \mathrm{dist}_{(M,X)}(0, \partial M) + \frac{\rho}{n}$.

Rough sketch of the proof

1. Split ∂M in a family of Jordan arcs $\alpha_1, \ldots, \alpha_k$ so that $\forall P \in \alpha_i$,

$$\rho - \xi < \|y\| \quad \forall y \in X(P) + \mathbf{\Pi}_i, \tag{1}$$

$$\|X(P) - y\| > \frac{\rho}{n} \quad \forall y \in X(P) + \mathbf{\Pi}_i \text{ with } \|y\| \geq \rho + \frac{1}{n^2}, \tag{2}$$

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2. Consider M_0 a simply connected compact region with $M \subset (M_0)^\circ \subset M_0 \subset \mathbb{D}$ and pairwise disjoint Jordan arcs $\gamma_1, \ldots, \gamma_k$ such that $\gamma_i \subset M_0 - M^\circ$ connects $\mathbf{P}_i$ and $\mathbf{Q}_i \in \partial M_0$, and $S = M \bigcup (\cup_{i=1}^k \gamma_i)$ is an admissible compact set on $\mathbb{D}$.

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3. Extend X to a generalized null curve $X : S \to \mathbb{C}^3$ so that

- On the first half of γ_i, the projection of X in the direction of $X(\mathbf{P}_i)$ has length $> \rho/n$, and X satisfies (1) and (2).
- X on the second half of γ_i is a segment in the direction of $X(\mathbf{P}_i)$ and

$$\rho + \frac{1}{n^2} < \|y\| \quad \forall y \in X(\mathbf{Q}_i) + \mathbf{\Pi}_i.$$

- The coordinate of X in the direction of $X(\mathbf{P}_i)$ does all the work on $\gamma_i \cup \alpha_i \cup \gamma_{i+1}$ at this moment.

4. Approximate $X : S \to \mathbb{C}^3$ by a null curve $X_0 : M_0 \to \mathbb{C}^3$ satisfying the same properties as X on S.

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5. Label Ω_i as the closed disc in M_0 bounded by α_i, γ_i, γ_{i+1} and a piece of ∂M_0. Consider K_i a proper compact disc on $\Omega_i - \gamma_i \cup \alpha_i \cup \gamma_{i+1}$ so that just the coordinate of X_0 in the direction of $X(\mathbf{P}_i)$ does all the work on $\Delta_i = \overline{\Omega_i - K_i}$.

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6. Consider a generalized null curve $Z : \overline{M_0 - \Delta_i} \to \mathbb{C}^3$ such that

- $Z|_{\overline{M_0 - \Omega_i}} = X_0$.
- $Z|_{K_i} = w + X_0$, where $w \in \mathbf{\Pi}_i$ is large enough so that

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Q.E.D.

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- $M = \mathbb{D} \longrightarrow M =$arbitrary topology:
  - Use the Approximation Lemma as a Bridge Principle for null curves to complicate the topology little by little.

- Antonio Alarcón, Isabel Fernández and Francisco J. López, *Complete minimal surfaces and harmonic functions*. Comment. Math. Helv., in press.

- Antonio Alarcón and Francisco J. López, *Minimal surfaces in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$*. Preprint 2009.

- Antonio Alarcón and Francisco J. López, *Null curves in $\mathbb{C}^3$ and Calabi-Yau conjectures*. Preprint 2009.

Thank you very much

for your kind attention!