Stability of capillary surfaces with planar or spherical boundary in the absence of gravity

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April 13^{th}2011

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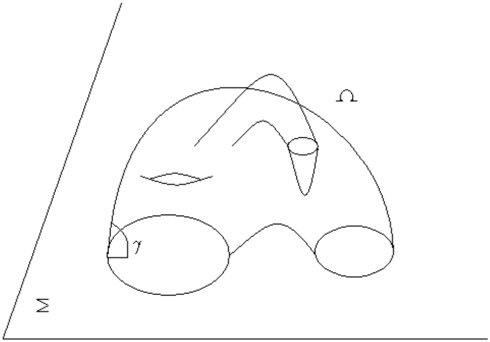
- Spherical boudary case: Construct a Killing vector field for the hyperbolic metric to show that if the centroid of the region bounded between the surface and the unit sphere is at the origin, the configuration cannot be stable.
- Corollary: New proof of Barbosa and Do Carmo's theorem for closed surfaces.

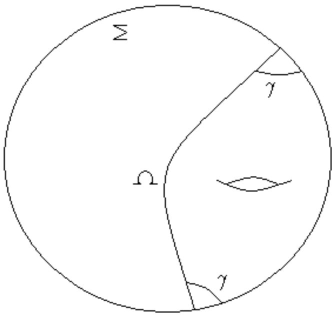
Problem 1

Figure: Immersed capillary surface "sitting" on a plane



Problem 2

Figure: Immersed capillary surface in a ball



Definition

The energy E of the above configuration is defined as

$$E = \sigma|\Omega| - \sigma\tau|\Sigma^{'}|$$

where σ is the surface tension and τ is the capillary constant.

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Let Ω be given by $x(D)$. An admissible variation of x is a differentiable map $\Phi : (-\epsilon, \epsilon) \times D \to \mathbb{R}^3$, such that $\Phi_t(p) = \Phi(t, p), p \in D$, is an immersion and $\Phi_0 = x$. Also the volume functional for planar Σ is given by

$$V(t) = \frac{1}{3} \int \int_D (\Phi_t \cdot \xi_t) dS_t$$

where ξ_t and dS_t are the unit outward normal and the surface element on $\Phi_t(D)$.

First Variation Formula

Let the normal component of Φ is given by $\phi = (Y \cdot \xi)$, where $Y = \frac{\partial \Phi}{\partial t}\Big|_{t=0}$. Also Let $d\sigma$ be the line element on the boundary Γ and dS be the surface element on Ω. The first variation formula for the energy of x in the direction of ϕ, subject to a volume constraint implies that

$$\partial(E)[\phi] \equiv \frac{d}{dt}E(t)\Big|_{t=0} = -2\int\int_D H\phi dS + \oint_{\partial D}(-\tau \csc\gamma + \cot\gamma)\phi d\sigma$$

$$\partial(V)[\phi] \equiv \frac{d}{dt}V(t)\Big|_{t=0} = \int\int_D \phi dS \equiv 0.$$

It follows that if we want Ω to be critical point for the energy, the mean curvature H must be constant, $\tau = \cos(\gamma)$ and γ must be constant.

Definition

A capillary surface is called weakly stable if the second variation is non negative for all admissible perturbations with normal components $\phi \neq 0$ and stable if the second variation is positive for all admissible perturbations.

Second Variation Formula

Following the above notation the formula for the Second Variation of E is

$$\partial^2(E)[\phi] \equiv \frac{d^2}{dt^2}E(t)\Big|_{t=0}$$
$$= \int\int_D [|\nabla\phi|^2 - (k_1^2 + k_2^2)\phi^2]dS$$
$$+ \oint_{\partial D} p\phi^2 d\sigma.$$

Here $\nabla\phi$ is the surface gradient of ϕ, k_1 and k_2 are the principal curvatures, and $p = K_\Omega \cot(\gamma) + K_\Sigma \csc(\gamma)$. Here K_Ω and K_Σ are the signed normal curvatures of Ω and Σ with respect to the boundary. Of course, the volume condition must be fulfilled.

Second Variation Formula

Using Green's first identity one gets

$$\partial^2 E = \int\int_D (-L\phi)\phi dS$$

$$+ \oint_{\partial D} (\phi_\nu + p\phi)\phi d\sigma$$

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Also

$$\partial V \equiv \int\int_D \phi ds = 0$$

The operator L is called the *Jacobi operator* (here Δ is the surface Laplacian on Ω).

Main Theorem

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(Planar boundary) There exists no stable capillary surface with planar boundary , that is immersed in $\mathbb{R}^3$ and having genus $g > 0$. The boundary is assumed to be embedded in Σ.

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In the genus zero case the only stable capillary surfaces with planar boundary are the spherical caps. Again no gravity is assumed anywhere.

The perturbation

Let

$$\Phi(x, t) = x + t\xi + Htx + ct\mathbf{k} + O(t^2)$$

where **k** is the vertical unit vector and c is a constant.

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$$\phi = 1 + H(x \cdot \xi) + c(\mathbf{k} \cdot \xi)$$

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$$\phi = 1 + H(x\cdot\xi) + c(\mathbf{k}\cdot\xi)$$

Need to determine c to keep the volume fixed.

Lemma

The volume constraint implies that $c = -\cos(\gamma)$, *i.e*

$$\phi = 1 + H(x \cdot \xi) - \cos(\gamma)(\mathbf{k} \cdot \xi)$$

in order to keep the volume fixed.

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Proof:

$$0 = \int\int_D \phi dS = \int\int_D (1 + H(x \cdot \xi) - c(\mathbf{k} \cdot \xi))dS$$
$$= |\Omega| + H\int\int_D (x \cdot \xi)dS + c\int\int_D (\mathbf{k} \cdot \xi)dS$$

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We have

$$\int\int_D (\mathbf{k} \cdot \xi)dS = |\Sigma^{'}|$$

Using conformal coordinates one has

$$\int\int_D H(x\cdot\xi)dS = \frac{1}{2}\int\int_D (x\cdot\Delta x)dS$$
$$= -\frac{1}{2}\int\int_D |\nabla x|^2 dS + \frac{1}{2}\oint_{\partial D}(x\cdot x_\nu)d\sigma$$

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$$|\nabla x|^2 = \frac{1}{E}((x_u\cdot x_u)+(x_v\cdot x_v)) = \frac{1}{E}(E+E) = 2$$

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Also one has

$$\frac{1}{2}\oint_{\partial D}(x\cdot x_\nu)d\sigma = \frac{\cos(\gamma)}{2}\oint_{\partial D}(x\cdot \mathbf{n})d\sigma = \cos(\gamma)|\Sigma'|$$

So far we have

$$0 = |\Omega| + H \int\int_D (x \cdot \xi) dS + c \int\int_D (\mathbf{k} \cdot \xi) dS$$

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The above four formulae imply that

$$c = -\cos(\gamma)$$

and

$$\phi = 1 + H(x \cdot \xi) - \cos(\gamma)(\mathbf{k} \cdot \xi)$$

Zero boundary term

Lemma

For $\phi = 1 + H(x \cdot \xi) - \cos(\gamma)(\mathbf{k} \cdot \xi)$ *on the boundary curve* Γ *we have*

$$\phi_\nu + p\phi = 0$$

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therefore

$$\partial^2 E = \int \int_D (-L\phi)\phi dS.$$

$$L\phi = L1 + L(H(x \cdot \xi)) - \cos(\gamma) L(\mathbf{k} \cdot \xi) = k_1^2 + k_2^2 + HL(x \cdot \xi)$$

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Taking this into account we have

$$L\phi = k_1^2 + k_2^2 - 2H^2 = k_1^2 + k_2^2 - \frac{(k_1+k_2)^2}{2} = \frac{(k_1-k_2)^2}{2}$$

therefore

$$\int\int_D (-L\phi)\phi dS = -\int\int_D \frac{(k_1-k_2)^2}{2} dS - \int\int_D \frac{(k_1-k_2)^2}{2}(H(x\cdot\xi) - \cos(\gamma)(\mathbf{k}\cdot\xi))dS$$

Rewriting the previous formula using theorems from differential geometry and some other facts we arrive at

$$\partial^2 E = -\int\int_D \frac{(k_1 - k_2)^2}{2} dS \\ - \oint_{\partial D} \cos(\gamma)[2H^2(x \cdot \mathbf{n}) + 2Hsin(\gamma)]d\sigma \\ + \oint_{\partial D} \cos(\gamma)[\sin(\gamma) k_\Gamma H(x \cdot \mathbf{n}) + \sin^2(\gamma) k_\Gamma]d\sigma$$

Auxiliary lemma

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- *(i)*

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- *(iii)*

$$\left| \oint_{\partial D} k_\Gamma \, d\sigma \right| \leq 2\pi d$$

- *(iv)*

$$\sin(\gamma)|\Gamma| = -2H|\Sigma'|$$

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$$\frac{(k_1 - k_2)^2}{2} = 2H^2 - 2K$$

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The auxiliary lemma and the above fact imply that

$$\partial^2 E = -2 \int\int_D H^2 dS + 2 \int\int_D K dS + \cos(\gamma)[2H^2|\Sigma'| + \sin^2(\gamma) \oint_{\partial D} k_\Gamma d\sigma].$$

Now Gauss-Bonnet theorem and again the auxiliary lemma imply that:

$$\partial^2 E = -2 \int\int_D H^2 dS + 4\pi\chi(\Omega) - 2 \oint_{\partial D} k_g d\sigma$$
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$$\leq -2H^2[|\Omega| - \cos\gamma|\Sigma'|] + 4\pi(2 - 2g - d)$$
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$$- 2\pi d[2 - 2|\cos\gamma| - |\cos\gamma|\sin^2\gamma].$$

We get

$$\partial^2 E \leq -2H^2[|\Omega| - \cos\gamma|\Sigma'|] + 4\pi(2-2g) \\ - 2\pi d[2 - 2|\cos\gamma| - |\cos\gamma|\sin^2\gamma]$$

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which yields that for $g > 0$

$$\partial^2 E < 0.$$

Let z be the complex coordinate in $\mathbb{R}^2$. The LFT's that map the unit disc into itself are of the form

$$w = e^{i\theta}\frac{z-\alpha}{1-z\bar{\alpha}}$$

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with $t \in (-1, 1)$.

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This is a family of hyperbolic LFT's (two fixed points $z = \pm 1$) and $w_0 = Id$. To see that we normalize to make the determinant one and observe that the square of the trace is bigger than 4.

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Rotating the above vector field about the x-axis we obtain a vector field F on $\mathbb{R}^3$

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Rotating the above vector field about the x-axis we obtain a vector field F on $\mathbb{R}^3$

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This is a conformal vector field (also Killing for the hyperbolic metric), i.e. it is the derivative at zero of a family of conformal mappings $\Phi_t(x) : \mathbb{R}^3 \to \mathbb{R}^3$ with $\Phi_0(x) = Id$. The family of conformal mappings is also a Lie group.

Theorem

$F = < F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) >^T$ is a vector field on $\mathbb{R}^3$, and let

$$DF = \begin{bmatrix} \frac{DF_1}{dx} & \frac{DF_1}{dy} & \frac{DF_1}{dz} \\ \frac{DF_2}{dx} & \frac{DF_2}{dy} & \frac{DF_2}{dz} \\ \frac{DF_3}{dx} & \frac{DF_3}{dy} & \frac{DF_3}{dz} \end{bmatrix}$$

be the differential of F. Then, F is conformal if and only if

$$DF + DF^T = \lambda(x,y,z)Id$$

where $\lambda(x,y,z)$ is a scalar function.

Properties of $F =< 1 - x^2 + y^2 + z^2, -2xy, -2xz >^T$

- $\partial\gamma = \phi_\nu + p\phi$ which implies that for conformal F the boundary integral in $\partial^2 E$ is zero.

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- F is that it is tangent to $\mathbb{S}^2$.
- $\phi = (F \cdot \xi) = (\mathbf{i} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])$.

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- $\phi = (F \cdot \xi) = (\mathbf{i} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])$.
- $\partial V = \int \int_D \phi dS = \int \int_D (F \cdot \xi) dS = -6x_0 Vol(T)$ where $< x_0, y_0, z_0 >$ is center of gravity of T.

$$\phi = (\mathbf{i} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])$$
$$\psi = (\mathbf{j} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])$$
$$\eta = (\mathbf{k} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])$$

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Lemma

$$L\phi = 4(\mathbf{i} \cdot [\xi + H\bar{x}])$$
$$L\psi = 4(\mathbf{j} \cdot [\xi + H\bar{x}])$$
$$L\eta = 4(\mathbf{k} \cdot [\xi + H\bar{x}])$$

$$Q_{11} = -4 \int\int_D (\mathbf{i} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])(\mathbf{i} \cdot [\xi + H\bar{x}])dS$$

$$Q_{22} = -4 \int\int_D (\mathbf{j} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])(\mathbf{j} \cdot [\xi + H\bar{x}])dS$$

$$Q_{33} = -4 \int\int_D (\mathbf{k} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])(\mathbf{k} \cdot [\xi + H\bar{x}])dS.$$

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$$Q_{22} = -4 \int\int_D (\mathbf{j} \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])(\mathbf{j} \cdot [\xi + H\bar{x}])dS$$

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Want to show $Q_{11} + Q_{22} + Q_{33} \leq 0$.

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Lemma

$$\Delta|\bar{x}|^2 = 4(1 + H(\bar{x} \cdot \xi)).$$

Observing that

$$(\mathbf{i} \cdot \xi)^2 + (\mathbf{j} \cdot \xi)^2 + (\mathbf{k} \cdot \xi)^2 = |\xi|^2 = 1$$

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we get

$$\sum_{i=1}^{3}Q_{ii}=-4\int\int_D[H(1-|\bar{x}|^2)(\bar{x}\cdot\xi)+1+|\bar{x}|^2-2(\bar{x}\cdot\xi)^2]dS.$$

Next, we estimate the integrand in the previous page. From *Cauchy-Schwarz* inequality it follows that $(\bar{x} \cdot \xi)^2 \leq |\bar{x}|^2 |\xi|^2 = |\bar{x}|^2$, therefore

$$1 + |\bar{x}|^2 - 2(\bar{x} \cdot \xi)^2 \geq 1 + |\bar{x}|^2 - 2|\bar{x}|^2 = 1 - |\bar{x}|^2.$$

Notice that $1 - |\bar{x}|^2 \geq 0$ since $\bar{x}$ represents the surface Ω which lies entirely in the unit ball B.

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Notice that $1 - |\bar{x}|^2 \ge 0$ since $\bar{x}$ represents the surface Ω which lies entirely in the unit ball B.
Using the previous lemma

$$\begin{aligned} -\sum_{i=1}^{3} Q_{ii} &\ge 4\int\int_D [H(1-|\bar{x}|^2)(\bar{x}\cdot\xi) + 1 - |\bar{x}|^2] dS \\ &= 4\int\int_D [(1-|\bar{x}|^2)(1 + H(\bar{x}\cdot\xi)) dS \\ &= \int\int_D (1-|\bar{x}|^2)\Delta|\bar{x}|^2 dS. \end{aligned}$$

Apply *Green's First Identity* and also the fact that $1 - |\bar{x}|^2 = 0$ on $\partial\Omega \subseteq \mathbb{S}^2$.

$$
\begin{aligned}
-\sum_{i=1}^{3} Q_{ii} &\geq \int\int_D (1 - |\bar{x}|^2)\Delta|\bar{x}|^2 dS \\
&= -\int\int_D (\nabla(1 - |\bar{x}|^2) \cdot \nabla|\bar{x}|^2) dS \\
&= \int\int_D (\nabla|\bar{x}|^2 \cdot \nabla|\bar{x}|^2) dS \\
&= \int\int_D |\nabla|\bar{x}|^2|^2 dS \geq 0.
\end{aligned}
$$

Therefore $\sum_{i=1}^{3} Q_{ii} < 0$ for nontrivial $\bar{x}$ satisfing the centroid condition.

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This method proves Barbosa and Do Carmo's theorem. One need just to put the centroid at the origin, which is possible since there is no boundary.

Thank you for your patience!