On holomorphic embeddings of Riemann surfaces into $\mathbb{C}^2$

The Bell-Narasimhan Conjecture

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Stein manifolds

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Definition (Stein 1951)

A complex manifold $\mathbf{S}$ of complex dimension n is said to be a **Stein manifold** iff the following two conditions hold:

- $\mathbf{S}$ is holomorphically convex, i.e.,

$$\hat{K} := \left\{ z \in \mathbf{S} \;\middle|\; |f(z)| \leq \sup_K |f| \;\; \forall f : \mathbf{S} \to \mathbb{C} \text{ holomorphic} \right\}$$

  is compact for all compact $K \subset \mathbf{S}$. Equivalently, if $E \subset \mathbf{S}$ is unbounded then there exists $f : \mathbf{S} \to \mathbb{C}$ holomorphic such that $f|_E$ is unbounded.

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- $\mathbf{S}$ is holomorphically separable, i.e., for any $z, w \in \mathbf{S}$, $z \neq w$, there exists $f : \mathbf{S} \to \mathbb{C}$ holomorphic with $f(z) \neq f(w)$.

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  is compact for all compact $K \subset \mathbf{S}$. Equivalently, if $E \subset \mathbf{S}$ is unbounded then there exists $f : \mathbf{S} \to \mathbb{C}$ holomorphic such that $f|_E$ is unbounded.
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- Roughly speaking, a Stein manifold $\mathbf{S}$ is a complex manifold that carries many holomorphic functions

$$\mathbf{S} \to \mathbb{C}.$$

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Theorem (Runge)

Let $\mathscr{R}$ be an open Riemann surface, let $K \subset \mathscr{R}$ be a compact set such that $\mathscr{R} - K$ has no bounded connected components (i.e., K is Runge), let $\varepsilon > 0$ and let $f : K \to \mathbb{C}$ holomorphic.
Then there exists $g : \mathscr{R} \to \mathbb{C}$ holomorphic such that

$$|f(z) - g(z)| < \varepsilon \ \forall z \in K.$$

Stein manifolds

- Roughly speaking, Stein manifolds are the complex manifolds whose function theory is similar to that of domains in $\mathbb{C}$.

Theorem (Weierstrass Theorem)

On a discrete subset of a domain Ω in $\mathbb{C}$, one can prescribe the values of a holomorphic function on Ω.

Theorem (Cartan Extension Theorem)

If $\mathbf{T}$ *is a closed complex submanifold of a Stein manifold* $\mathbf{S}$, *then every holomorphic function on* $\mathbf{T}$ *extends to a holomorphic function on* $\mathbf{S}$.

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Theorem (Oka-Weil Approximation Theorem)

If a compact subset K of a Stein manifold $\mathbf{S}$ is holomorphically convex, i.e., $K = \hat{K}$, then every holomorphic function on K can be approximated uniformly on K by holomorphic functions on $\mathbf{S}$.

- A compact subset K of an open Riemann surface is holomorphically convex if and only if it is Runge.
  In higher dimensions, holomorphic convexity is much more subtle; in particular, it is not a topological property.

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Conjecture (Forster 1967)

Any Stein manifold of complex dimension

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admits a proper holomorphic embedding into $\mathbb{C}^N$ (and a proper holomorphic immersion into $\mathbb{C}^{N-1}$) with

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- Eliashberg-Gromov 1992, Schürmann 1997 The Forster conjecture holds.

Case $n = 1$

- A one-dimensional Stein manifold is the same thing as an open Riemann surface.
- For $n = 1$ the Forster conjecture would predict that any open Riemann surface admits a proper holomorphic embedding into $\mathbb{C}^2$, but the proof in the cited papers breaks down in this case.

Case $n = 1$

- A one-dimensional Stein manifold is the same thing as an open Riemann surface.
- For $n = 1$ the Forster conjecture would predict that any open Riemann surface admits a proper holomorphic embedding into $\mathbb{C}^2$, but the proof in the cited papers breaks down in this case.
- The main problem is that self-intersections of an immersed complex curve in $\mathbb{C}^2$ are stable under deformations.

The Conjecture

Conjecture (Bell-Narasimhan 1990)

Any open Riemann surface admits a proper holomorphic embedding into $\mathbb{C}^2$.

What is known

- Remmert 1956, Narasimhan 1960, Bishop 1961 Any open Riemann surface admits
  - a proper holomorphic embedding into $\mathbb{C}^3$, and
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- Globevnik-Stensønes 1995 Any finitely connected planar domain without isolated boundary points embeds.
- Černe-Forstnerič 2002 Any open orientable surface of **finite topology** admits a complex structure that embeds.

(Slow development.)

Prof. Franc Forstnerič

Forstnerič will visit Granada in late November'11

Wold's idea: Fatou-Bieberbach domains

Definition

A domain $\Omega \subsetneq \mathbb{C}^2$ is said to be a Fatou-Bieberbach domain if there exists a biholomorphism

$$\psi : \Omega \to \mathbb{C}^2.$$

In this case, ψ is said to be a Fatou-Bieberbach map.

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Wold's idea 2006 Let $\overline{\mathscr{R}}$ be a compact Riemann surface with non-empty boundary and let $\psi : \Omega \to \mathbb{C}^2$ be a Fatou-Bieberbach map. If there exists a holomorphic embedding

$$\mathbf{X} : \overline{\mathscr{R}} \to \mathbb{C}^2$$

with

$$\mathbf{X}(\mathscr{R}) \subset \Omega \text{ and } \mathbf{X}(\partial \mathscr{R}) \subset \partial \Omega,$$

then $\psi \circ \mathbf{X} : \mathscr{R} \to \mathbb{C}^2$ is a proper holomorphic embbeding.

Wold's idea: Fatou-Bieberbach domains

- Let $\mathscr{R} = \mathbb{D}$. Let $\Omega \subset \mathbb{C}^2$ be a Runge Fatou-Bieberbach domain. Let U be a connected component of $\Omega \cap (\mathbb{C} \times \{z_0\})$, $U \neq (\mathbb{C} \times \{z_0\})$. Since Ω is Runge then U is simply connected. Let $\mathbf{X} : \mathbb{D} \to U$ be a biholomorphism and let $\psi : \Omega \to \mathbb{C}^2$ be a Fatou-Bieberbach map.
  Then $\psi \circ \mathbf{X} : \mathbb{D} \to \mathbb{C}^2$ is a proper holomorphic embedding.

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- Wold 2006 Under some (very technical) conditions, there exists a holomorphic embedding

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where Ω is a Fatou-Bieberbach domain.

  - Any finitely-connected planar domain embeds.
  - Any subset of a torus with two boundary components embeds.

Forstnerič-Wold technique

Theorem (Forstnerič-Wold 2009)

Let $\mathscr{R}$ *be a* **bordered Riemann surface** *(i.e., the interior of a compact Riemann surface with non-empty boundary consisting of a finite family of smooth Jordan curves) that admits a (non-necessarily proper) holomorphic embedding into* $\mathbb{C}^2$*.*
Then $\mathscr{R}$ *embeds.*

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Let $\mathscr{R}$ be a **bordered Riemann surface** *(i.e., the interior of a compact Riemann surface with non-empty boundary consisting of a finite family of smooth Jordan curves) that admits a (non-necessarily proper) holomorphic embedding into $\mathbb{C}^2$. Then $\mathscr{R}$ embeds.*

1. Consider a (non-proper) holomorphic embedding $\mathbf{X} : \mathscr{R} \to \mathbb{C}^2$.
2. Modify the embedding $\mathbf{X}$ so that $\mathbf{X}(\mathscr{R})$ satifies the Wold conditions. Then

$$\mathbf{X}(\mathscr{R}) \subset \Omega \text{ and } \mathbf{X}(\partial\mathscr{R}) \subset \partial\Omega,$$

   where $\Omega \subset \mathbb{C}^2$ is a Fatou-Bieberbach domain.
3. Compose the modified $\mathbf{X}$ with a Fatou-Bieberbach map $\psi : \Omega \to \mathbb{C}^2$.

Open Questions

- The embedding problem naturally decouples in the following two problems:

Question

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Does any open Riemann surface admit a (non-necessarily proper) holomorphic embedding into $\mathbb{C}^2$*?*

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If an open Riemann surface admit a (non-necessarily proper) holomorphic embedding into $\mathbb{C}^2$*, does it embed?*

Arbitrary topology

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- Idea: Modify the embedding $\mathbf{X}$ but also the Riemann surface $\mathscr{R}$.

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Theorem (Alarcón-López 2011)

Let $\mathbf{N}$ be an open Riemann surface.
Then there exists an open domain $\mathbf{M} \subset \mathbf{N}$ such that

- $\mathbf{M} \overset{hom}{\cong} \mathbf{N}$, *and*
- $\mathbf{M}$ *embeds.*

In particular, any open orientable surface $\mathscr{S}$ admits a complex structure $\mathscr{C}$ such that the Riemann surface $\mathscr{R} := (\mathscr{S}, \mathscr{C})$ embeds.

Lemma

Let $\mathbf{N}$ *be an open Riemann surface, let* $\mathscr{R} \subset \mathbf{N}$ *be a Runge compact region, let* $\mathbf{X} : \mathscr{R} \to \mathbb{C}^2$ *be a holomorphic embedding and let* $r > 0$ *such that*

$$\mathbf{X}(\partial\mathscr{R}) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(r).$$

Then for any $\varepsilon > 0$ *and any* $\rho > r$ *there exists a Runge compact region* $\mathscr{S} \subset \mathbf{N}$ *and a embedding* $\mathbf{Y} : \mathscr{S} \to \mathbb{C}^2$ *with*

- $\mathscr{S} \overset{hom}{\cong} \mathscr{R}$, $\mathscr{R} \subset \mathscr{S}^\circ$,
- $\|\mathbf{Y}(z) - \mathbf{X}(z)\| < \varepsilon \ \forall z \in \mathscr{R}$,
- $\mathbf{Y}(\partial\mathscr{S}) \subset \mathbb{C}^2 - \overline{\mathbb{B}}(\rho)$, *and*
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- Applying the lemma recursively one can find a Riemann surface $\mathscr{R}_\infty$ homeomorphic to $\mathscr{R}^\circ$ that embeds.

Tools

- How can one complicate the topology?

Theorem (Mergelyan-Bishop 1951-1958)

Let $\mathscr{R}$ be an open Riemann surface, let $K \subset \mathscr{R}$ be a Runge compact set, let $\varepsilon > 0$ and let $f : K \to \mathbb{C}$ continuous with $f|_{K^\circ} : K^\circ \to \mathbb{C}$ holomorphic.
Then there exists $g : \mathscr{R} \to \mathbb{C}$ holomorphic such that

$$|f(z) - g(z)| < \varepsilon \ \forall z \in K.$$

Tools

- Given an open Riemann surface $\mathscr{R}$, a compact subset $S \subset \mathscr{R}$ is said to be admissible iff
  - S is Runge,
  - $R_S := \overline{S^\circ}$ consists of a finite collection of pairwise disjoint compact regions in $\mathscr{R}$,
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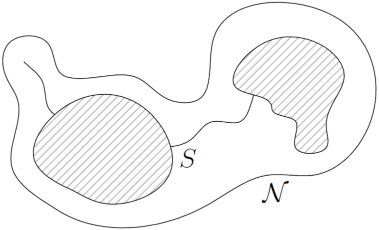

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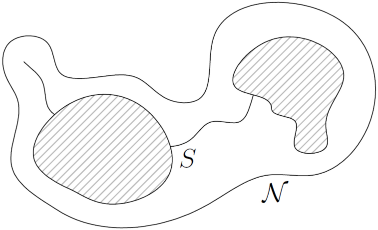

- Let $f : S \to \mathbb{C}$ be a continuous function such that $f|_{R_S} : R_S \to \mathbb{C}$ is holomorphic and let $\varepsilon > 0$, then there exists $g : S \to \mathbb{C}$ holomorphic with $|f(z) - g(z)| < \varepsilon \ \forall z \in S$.

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- Mergelyan's Theorem $\Rightarrow$ One can add handles/ends to a complex curve of $\mathbb{C}^2$.
- Ferrer-Martín-Meeks 2009 Every (open orientable) topological surface can be obtained in a recursive process by adding either a handle or an end in each step.
- Lemma + Mergelyan + Ferrer-Martín-Meeks $\Rightarrow$ Theorem

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Steps of the proof of the Lemma

1. Expose boundary points $\{b_j\} \subset \partial\mathscr{R}$ with respect to $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$, $\pi_1(z_1, z_2) = z_1$. Be careful with $\overline{\mathbb{B}(r)}$ and $\overline{\mathbb{B}(\rho)}$.

Steps of the proof of the Lemma

Definition

Let $\mathscr{R} \subset \mathbf{N}$ be a Riemann surface possibly with boundary, and let $X : \mathscr{R} \to \mathbb{C}^2$ be a proper holomorphic embedding. A point $p = (p_1, p_2)$ of the complex curve $\Sigma := X(\mathscr{R})$ is said to be *exposed with respect to* π_1 if the complex line

$$\Lambda_p = \pi_1^{-1}(\pi_1(p)) = \{(p_1, w) \mid w \in \mathbb{C}\}$$

intersects Σ only at p and this intersection is transverse, that is to say, $\Lambda_p \cap \Sigma = \{p\}$ and $T_p\Lambda_p \cap T_p\Sigma = \{0\}$.

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2. Blow up the exposed points $\{b_j\}$ with respect to $\pi_2 : \mathbb{C}^2 \to \mathbb{C}$, $\pi_2(z_1, z_2) = z_2$. Compose $\mathbf{X}$ with the map

$$\mathbf{g} : \mathbb{C}^2 \to \mathbb{C} \times \overline{\mathbb{C}}$$

$$\mathbf{g}(z, w) = \left(z \, , \; w + \sum \frac{\alpha_j}{z - \pi_1(\mathbf{X}(b_j))} \right).$$

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(a) $(\mathbf{g} \circ \mathbf{X})(\partial \mathscr{R})$ admits an exhaustion by compact regions $K_1 \subset K_2 \subset \dots$ which are polynomially convex in $\mathbb{C}^2$.

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(b) For any compact polynomially convex set $K \subset \mathbb{C}^2 - \partial \left[(\mathbf{g} \circ \mathbf{X})(\partial \mathscr{R}) \right]$, the compact set $K \cup K_j$ is also polynomially convex in $\mathbb{C}^2$ for all large $j \in \mathbb{N}$.

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(c) For every compact polynomially convex set K contained in $\mathbb{C}^2 - \partial\left[(\mathbf{g}\circ\mathbf{X})(\partial\mathscr{R})\right]$, and for every pair of numbers $\xi > 0$ and $R > 0$ there exists a holomorphic automorphism φ of $\mathbb{C}^2$ such that

$$\|\varphi - \mathrm{Id}\| < \xi \text{ on } K \quad \text{and} \quad \varphi\left(\partial\left[(\mathbf{g}\circ\mathbf{X})(\partial\mathscr{R})\right]\right) \subset \mathbb{C}^2 - \overline{\mathbb{B}(R)}.$$

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(b) For any compact polynomially convex set $K \subset \mathbb{C}^2 - \partial\left[(\mathbf{g} \circ \mathbf{X})(\partial \mathscr{R})\right]$, the compact set $K \cup K_j$ is also polynomially convex in $\mathbb{C}^2$ for all large $j \in \mathbb{N}$.

(c) For every compact polynomially convex set K contained in $\mathbb{C}^2 - \partial\left[(\mathbf{g} \circ \mathbf{X})(\partial \mathscr{R})\right]$, and for every pair of numbers $\xi > 0$ and $R > 0$ there exists a holomorphic automorphism φ of $\mathbb{C}^2$ such that

$$\|\varphi - \mathrm{Id}\| < \xi \text{ on } K \quad \text{and} \quad \varphi\left(\partial\left[(\mathbf{g} \circ \mathbf{X})(\partial \mathscr{R})\right]\right) \subset \mathbb{C}^2 - \overline{\mathbb{B}(R)}.$$

4. Apply **(c)** to $K := \overline{\mathbb{B}(r)} \cup K_j$ for a large enough j and $R > \rho$, and compose $\mathbf{g} \circ \mathbf{X}$ with the resulting φ.

Steps of the proof of the Lemma

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4. Apply **(c)** to $K := \overline{\mathbb{B}(r)} \cup K_j$ for a large enough j and $R > \rho$, and compose $\mathbf{g}\circ\mathbf{X}$ with the resulting φ. Shrink $\mathscr{R}$ suitably.

On holomorphic embeddings of Riemann surfaces into $\mathbb{C}^2$

The Bell-Narasimhan Conjecture

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