Harmonic diffeomorphisms between domains in $\mathbb{S}^2$

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Joint work with R. Souam

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Harmonic maps between Riemannian manifolds

- Let $\mathbf{M} = (M, g)$ and $\mathbf{N} = (N, h)$ be smooth Riemannian manifolds.
- Given a smooth map $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ and a domain $\Omega \subset \mathbf{M}$ with piecewise smooth boundary,

$$E_\Omega(\mathbf{f}) = \frac{1}{2}\int_\Omega |d\mathbf{f}|^2 dV_g$$

  is said to be the energy of $\mathbf{f}$ over Ω.
- A smooth map $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ is said to be harmonic if it is a critical point of the energy functional.

Well known facts

- Harmonicity of a map from a Riemann surface is well defined. If $\mathbf{M} = (M^2, g)$ is a surface, then the energy integral of a smooth map $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ is invariant under conformal changes of the metric g, and thus so is the harmonicity of $\mathbf{f}$.

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- Harmonicity of a map into a Riemannian surface $\mathbf{N} = (N^2, h)$ is highly sensible under conformal changes of the metric h.
- An isometric immersion $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ is harmonic if and only if $\mathbf{f}(\mathbf{M})$ is a minimal submanifold of $\mathbf{N}$.

Existence or not of harmonic diffeomorphisms

- Liouville There is no non-constant harmonic map $\mathbb{C} \to \mathbb{D}$, with the euclidean metric.
- Heinz 1952 There is no harmonic diffeomorphism $\mathbb{D} \to \mathbb{C}$ with the euclidean metric.
  Bernstein theorem: An entire minimal graph over the euclidean plane is a plane.

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Conjecture (Schoen-Yau 1985)

There is no proper harmonic map $\mathbb{D} \to \mathbb{C}$. In particular, no hyperbolic minimal surface in $\mathbb{R}^3$ properly projects into a plane.

Theorem (A-López 2009)

Any open Riemann surface admits a conformal minimal immersion in $\mathbb{R}^3$ properly projecting into a plane.

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Question (Schoen-Yau 1985)

Are Riemannian surfaces which are related by a harmonic diffeomorphism quasiconformally related?

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Are Riemannian surfaces which are related by a harmonic diffeomorphism quasiconformally related?

No!

Theorem (Markovic 2002)

There is a pair of Riemannian surfaces of infinite topology which are related by a harmonic diffeomorphism but not by a quasiconformal diffeomorphism.

But...

Theorem (Markovic 2002)

The answer to the question by Schoen and Yau is positive in the finite topology case, under some additional assumptions.

Existence or not of harmonic diffeomorphisms

Conjecture (Schoen-Yau 1985)

There is no harmonic diffeomorphism from $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}$.

- Collin-Rosenberg 2010 There exists an entire minimal graph Σ over $\mathbb{H}$ in the Riemannian product $\mathbb{H} \times \mathbb{R}$ with the conformal type of $\mathbb{C}$.
  In particular, the vertical projection $\Sigma \to \mathbb{H}$ is a harmonic diffeomorphism from $\mathbb{C}$ into $\mathbb{H}$.

Existence or not of harmonic diffeomorphisms

- A domain in the Riemann sphere $\overline{\mathbb{C}}$ is said to be a circular domain if every connected component of its boundary is a circle.

Theorem (A-Souam 2011)

(i) *For any* $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$, *and any subet* $\{p_1, \ldots, p_{\mathfrak{m}}\} \in \mathbb{S}^2$ *there exist a circular domain* $\mathbf{U} \subset \overline{\mathbb{C}}$ *and a harmonic diffeomorphism* $\mathbf{U} \to \mathbb{S}^2 - \{p_1, \ldots, p_m\}$.

(ii) *There exists no harmonic diffeomorphism* $\mathbb{D} \to \mathbb{S}^2 - \{p\}$, $p \in \mathbb{S}^2$.

(iii) *For any* $\mathfrak{m} \in \mathbb{N}$, *any subset* $\{z_1, \ldots, z_{\mathfrak{m}}\} \subset \overline{\mathbb{C}}$ *and any pairwise disjoint closed discs* $D_1, \ldots, D_{\mathfrak{m}}$ *in* $\mathbb{S}^2$, *there exists no harmonic diffeomorphism* $\overline{\mathbb{C}} - \{z_1, \ldots, z_{\mathfrak{m}}\} \to \mathbb{S}^2 - \cup_{j=1}^{\mathfrak{m}} D_j$.

(i)

Existence of harmonic diffeomorphisms

$$\mathbf{U} \to \mathbb{S}^2 - \{p_1, \ldots, p_{\mathfrak{m}}\}.$$

Strategy

Similarly to Collin-Rosenberg,

- our strategy to show the harmonic diffeomorphism of Item (i) consists of constructing a maximal graph Σ over $\mathbb{S}^2 - \{p_1, \ldots, p_\mathfrak{m}\}$ in the Lorentzian manifold $\mathbb{S}^2 \times \mathbb{R}_1$, with the conformal structure of a circular domain. Then, the projection

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- our construction method is completely different and relies on the theory of maximal hypersurfaces in Lorentzian manifolds. More precisely, we proceed by solving Dirichlet problems.

Maximal graphs. Notation

- $\mathbb{M} = (\mathbb{M}, \langle \cdot, \cdot \rangle_{\mathbb{M}}) \equiv$ compact $\mathfrak{n}$-dimensional Riemannian manifold without boundary, $\mathfrak{n} \in \mathbb{N}$, $\mathfrak{n} \geq 2$.

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$$\langle \cdot, \cdot \rangle = \pi^*_{\mathbb{M}}(\langle \cdot, \cdot \rangle_{\mathbb{M}}) - \pi^*_{\mathbb{R}}(dt^2) = \langle \cdot, \cdot \rangle_{\mathbb{M}} - dt^2.$$

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- X^u, u is spacelike (i.e., induces a Riemannian metric on Ω) if and only if $|\nabla u| < 1$ on Ω.
- X^u, u is maximal if u is spacelike and H vanishes identically on Ω.

Maximal graphs and harmonic diffeomorphisms

- If $u : \Omega \to \mathbb{R}$ is maximal then

$$X^u : (\Omega, \langle \cdot, \cdot \rangle_u) \to (\mathbb{M} \times \mathbb{R}_1, \langle \cdot, \cdot \rangle)$$

is a harmonic map.

In particular

$$\mathrm{Id} : (\Omega, \langle \cdot, \cdot \rangle_u) \to (\Omega, \langle \cdot, \cdot \rangle_{\mathbb{M}})$$

is a harmonic diffeomorphism, and

$$u : (\Omega, \langle \cdot, \cdot \rangle_u) \to \mathbb{R}$$

is a harmonic function.

Maximal graphs and harmonic diffeomorphisms

- $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$.
- $\{p_1, \ldots, p_{\mathfrak{m}}\} \subset \mathbb{M}$.
- $\Omega = \mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\}$.

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- Is there a maximal graph over Ω in $\mathbb{M} \times \mathbb{R}_1$?

- If $\mathbb{M} = \mathbb{S}^2$, does such a graph have the conformal structure of a circular domain?

Maximal graphs over $\mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\}$ in $\mathbb{M} \times \mathbb{R}_1$

- $\mathfrak{A} = \{(p_i, t_i)\}_{i=1}^{\mathfrak{m}} \subset \mathbb{M} \times \mathbb{R}$ such that

$$|t_i - t_j| < \mathrm{dist}_{\mathbb{M}}(p_i, p_j) \quad \forall i, j \in \{1, \ldots, \mathfrak{m}\},\ i \neq j$$

(spacelike condition).

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- B_i^n, $(i, n) \in \{1, \ldots, \mathfrak{m}\} \times \mathbb{N}$, open disc in $\mathbb{M}$
  - ∂B_i^n smooth Jordan curve,
  - $\overline{B_i^n} \cap \overline{B_j^n} = \emptyset$ if $i \neq j$,
  - $\overline{B_i^{n+1}} \subset B_i^n$,
  - $\{p_i\} = \cap_{n \in \mathbb{N}} B_i^n$.

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- $\Delta_n = \mathbb{M} - \cup_{i=1}^{\mathfrak{m}} B_i^n$, $n \in \mathbb{N}$.

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- $t_i^n \in \mathbb{R}$, $\{t_i^n\}_{n \in \mathbb{N}} \to t_i$, $i = 1, \ldots, \mathfrak{m}$.

Maximal graphs over $\mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\}$ in $\mathbb{M} \times \mathbb{R}_1$

- $\varphi_n : \partial\Delta_n \to \mathbb{R}$

$$\varphi_n|_{\partial B_i^n} = t_i^n, \quad i = 1, \ldots, \mathfrak{m},$$

is ε_n-Lipschitz, $\varepsilon_n \in (0, 1)$.

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- Smoothing $\widetilde{\varphi}_n$, there exists a smooth spacelike function

$$\overline{\varphi}_n : \Delta_n \to \mathbb{R}$$

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$$\overline{\varphi}_n|_{\partial B_i^n} = t_i^n, \quad i = 1, \dots, \mathfrak{m}.$$

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- Gerhardt 1983 There exists a maximal function

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- $$\left.\begin{array}{l} \{u_n\}_{n\in\mathbb{N}} \text{ uniformly bounded} \\ |\nabla u_n| < 1 \text{ on } \Delta_n \end{array}\right\} \overset{\text{(Ascoli-Arzela)}}{\Longrightarrow}$$

- $\{u_n\}_{n\in\mathbb{N}}$ uniformly converges on compact sets of $\mathbb{M} - \{p_i\}_{i=1}^{\mathfrak{m}} = \cup_{n\in\mathbb{N}}\Delta_n$ to a Lipschitz function

$$\hat{u} : \mathbb{M} - \{p_i\}_{i=1}^{\mathfrak{m}} \to \mathbb{R}$$

with $\quad |\nabla \hat{u}| \leq 1$ a.e. in $\mathbb{M} - \{p_i\}_{i=1}^{\mathfrak{m}}$.

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- $\hat{u}$ extends to a Lipschitz function

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with $\quad |\nabla u| \leq 1$ a.e. in $\mathbb{M} - \{p_i\}_{i=1}^{\mathfrak{m}}$ and $\quad u(p_i) = t_i$ $\forall i = 1, \ldots, \mathfrak{m}$.

Maximal graphs over $\mathbb{M} - \{p_1, \ldots, p_{\mathfrak{m}}\}$ in $\mathbb{M} \times \mathbb{R}_1$

- Bartnik 1988 $\hat{u}$ is smooth (hence, a maximal function) except for a set of points

$$\Lambda \subset \mathbb{M} - \{p_i\}_{i=1}^{\mathfrak{m}},$$

$$\Lambda := \left\{p \in \mathbb{M} - \{p_i\}_{i=1}^{\mathfrak{m}} \mid (p, \hat{u}(p)) = \gamma(s_0) \text{ for some } 0 < s_0 < 1,\right.$$

where

$$\gamma : [0,1] \to \mathbb{M} \times \mathbb{R}_1$$

is a null geodesic such that

$$\gamma((0,1)) \subset X^{\hat{u}}(\mathbb{M} - \{p_i\}_{i=1}^{\mathfrak{m}})$$

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- Since $\mathfrak{A}$ satisfies the spacelike condition then $\Lambda = \emptyset$ and

$$\hat{u} : \mathbb{M} - \{p_1, \ldots, p_\mathfrak{m}\} \to \mathbb{R}$$

determines a maximal graph over $\mathbb{M} - \{p_1, \ldots, p_\mathfrak{m}\}$ in $\mathbb{M} \times \mathbb{R}_1$.

Maximal graphs over $\mathbb{M} - \{p_1, \ldots, p_\mathfrak{m}\}$ in $\mathbb{M} \times \mathbb{R}_1$

Theorem

Let $\mathbb{M}$ *be a compact Riemannian manifold, let* $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$, *and let*

$$\mathfrak{A} = \{(p_i, t_i)\}_{i=1}^{\mathfrak{m}} \subset \mathbb{M} \times \mathbb{R}$$

satisfying the spacelike condition.
Then there exists exactly one entire graph Σ *over* $\mathbb{M}$ *in* $\mathbb{M} \times \mathbb{R}_1$ *such that*

- $\mathfrak{A} \subset \Sigma$ *and*
- $\Sigma - \mathfrak{A}$ *is a spacelike maximal graph over* $\mathbb{M} - \{p_i\}_{i=1,\ldots,\mathfrak{m}}$.

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- *$\mathfrak{A} \subset \Sigma$ and*
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Maximal graphs over $\mathbb{M} - \{p_1, \ldots, p_\mathfrak{m}\}$ in $\mathbb{M} \times \mathbb{R}_1$

If $\mathbb{M}$ is a surface

$(\mathbb{M} - \{p_1, \ldots, p_\mathfrak{m}\}, \langle \cdot, \cdot \rangle_u)$ is conformally an open Riemann surface with the same genus as $\mathbb{M}$ and $\mathfrak{m}$ hyperbolic ends.

Corollary

Let $\mathbb{M}$ be a compact Riemannian surface, let $\mathfrak{m} \geq 2$ and let $\{p_1, \ldots, p_\mathfrak{m}\} \subset \mathbb{M}$. Then there exist an open Riemann surface $\mathscr{R}$ and a harmonic diffeomorphism $\mathscr{R} \to \mathbb{M} - \{p_1, \ldots, p_\mathfrak{m}\}$ such that every end of $\mathscr{R}$ is of hyperbolic type.

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If $\mathbb{M} = \mathbb{S}^2$, by Koebe's uniformization theorem,

Corollary

Let $\mathfrak{m} \in \mathbb{N}$, $\mathfrak{m} \geq 2$, and let $\{p_1, \ldots, p_{\mathfrak{m}}\} \subset \mathbb{S}^2$.
Then there exist a circular domain $\mathbf{U}$ *in $\overline{\mathbb{C}}$ and a harmonic diffeomorphism* $\mathbf{U} \to \mathbb{S}^2 - \{p_1, \ldots, p_{\mathfrak{m}}\}$.

(ii)

Non-existence of harmonic diffeomorphisms

$$\mathbb{D} \to \mathbb{S}^2 - \{p\}.$$

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Constant Gauss Curvature Surfaces

- S smooth simply-connected surface.
- $X : S \to \mathbb{R}^3$ immersion with constant Gauss curvature $K = 1$.
- II_X positive definite metric $\Rightarrow$ II_X induces on S a conformal structure, $\mathscr{S}$.
- $z = u + \imath v$ conformal parameter on $\mathscr{S}$.

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- Gálvez-Martínez 2000 The Gauss map $N : \mathscr{S} \to \mathbb{S}^2$ satisfies

$$X_u = N \times N_v \quad \text{and} \quad X_v = N \times N_u, \tag{1}$$

hence it is a harmonic local diffeomorphism.
Conversely, let $N : \mathscr{S} \to \mathbb{S}^2$ be a harmonic local diffeomorphism. Then the map $X : \mathscr{S} \to \mathbb{R}^3$ given by (1) is an immersion with constant Gauss curvature $K = 1$.

Constant Gauss Curvature Surfaces

- Gálvez-Hauswirth-Mira 2010

$$\begin{cases} I_X = \langle dX, dX \rangle_{\mathbb{R}^3} & = & Qdz^2 + 2\mu|dz|^2 + \overline{Q}d\overline{z}^2 \\ II_X = \langle dX, dN \rangle_{\mathbb{R}^3} & = & 2\rho|dz|^2 \\ III_X = \langle dN, dN \rangle_{\mathbb{R}^3} & = & -Qdz^2 + 2\mu|dz|^2 - \overline{Q}d\overline{z}^2, \end{cases}$$

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- Klotz 1980 There exists an immersion $Y : \mathscr{S} \to \mathbb{R}^3$ of constant Gauss curvature $K = 1$ such that $I_Y = III_X$, $II_Y = II_X$ and $III_Y = I_X$.

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(iii)

Non-existence of harmonic diffeomorphisms

$$\overline{\mathbb{C}} - \{z_1, \dots, z_{\mathfrak{m}}\} \to \mathbb{S}^2 - \cup_{j=1}^{\mathfrak{m}} D_j$$

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Proposition

Let $\mathscr{R}$ be a parabolic open Riemann surface, let N be an oriented Riemannian surface and let $\phi : \mathscr{R} \to N$ be a harmonic local diffeomorphism. Suppose that N has Gaussian curvature $K_N > 0$. Then ϕ is either holomorphic or antiholomorphic.

Non-existence, $\overline{\mathbb{C}} - \{z_1, \ldots, z_{\mathfrak{m}}\} \to \mathbb{S}^2 - \cup_{j=1}^{\mathfrak{m}} D_j$

- Assume that ϕ preserves orientation. Is ϕ holomorphic?

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- The Jacobian of ϕ, $J(\phi) = |\partial\phi|^2 - |\overline{\partial}\phi|^2 > 0$.
- By contradiction: If ϕ is not holomorphic, then the zeros of $|\overline{\partial}\phi|$ are isolated.

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- In the case when ϕ reverses orientation then a parallel argument gives that ϕ is antiholomorphic.

Harmonic diffeomorphisms between domains in $\mathbb{S}^2$

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Joint work with R. Souam

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