A complete bounded complex submanifold of $\mathbb{C}^2$

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Joint work with Francisco J. López

Variational problems and Geometric PDE's

Granada, June 21st 2013

Aim

The aim of this talk is to show that

> *any convex domain of $\mathbb{C}^2$ carries complete properly embedded complex curves.*

In particular,

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- Antonio Alarcón and Francisco J. López, *Complete bounded embedded complex curves in $\mathbb{C}^2$.* Preprint May 2013 (arXiv:1305.2118).

Introduction

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- Self-intersections of complex curves in $\mathbb{C}^2$ (generically, isolated double points) are stable under deformations.

Global problems: Properness

Conjecture (Forster 1967, Bell-Narasimhan 1990)

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- Remmert 1956, Narasimhan 1960, Bishop 1961 Every open Riemann surface admits
  - a proper holomorphic embedding into $\mathbb{C}^3$, and
  - a proper holomorphic immersion into $\mathbb{C}^2$.
- Forstnerič-Wold 2009-2011 If a *bordered* Riemann surface embeds in $\mathbb{C}^2$ then it properly embeds. Every *circled domain* in the Riemann sphere properly embeds.
- A-López 2011 The problem is purely complex analytic: there are no topological obstructions.

Global problems: Completeness

Question (Yang 1977)

Do there exist complete holomorphic embeddings $\mathbf{M}^k \to \mathbb{C}^n$, $k < n$, *which have a bounded image?*

Global problems: Completeness + boundedness

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Question

Given $k \in \mathbb{N}$, what is the smallest n?

Complex curves in $\mathbb{C}^2$

Question

What about the case $k = 1$ and $n = 2$? Do there exist complete bounded embedded complex curves in $\mathbb{C}^2$?

- **Main Problem:** Self-intersections of complex curves in $\mathbb{C}^2$ are stable under deformations.
- There are plenty of complete bounded immersed complex curves in $\mathbb{C}^2$:
  - Jones 1979 Simply-connected examples.
  - Martín-Umehara-Yamada 2009 Examples with finite topology.
  - A-López 2012 Examples with any given (possibly infinite) topology.
  - A-Forstnerič 2012 Examples normalized by any given bordered Riemann surface.

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	minimal surfaces $\mathbb{R}^3$	complex curves $\mathbb{C}^2$	null curves $\mathbb{C}^3$
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infinite genus	**?**	Our theorem $\exists$	AL-AF $\exists$

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Corollary

Let $k \in \mathbb{N}$. There exist

- *complete bounded embedded complex k-dimensional submanifolds of $\mathbb{C}^{2k}$, and*
- *complete bounded embedded complex k-dimensional submanifolds of $\mathbb{C}^{4k}$ with strongly negative holomorphic sectional curvature. (Topic in Yang's paper.)*

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Any convex domain $\mathcal{B} \subset \mathbb{C}^2$ carries complete properly embedded complex curves.

- Now it is the time to explain the primary idea of the proof!

Image completeness

- **Idea:** Take an already known immersed example and desingularize it (replacing every normal crossing in the curve by an embedded annulus).

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- Given $\mathbf{X}\colon \mathcal{R} \to \mathbb{C}^2$, we denote by $\mathrm{dist}_{\mathbf{X}(\mathcal{R})}$ the (intrinsic) induced Euclidean distance in $\mathbf{X}(\mathcal{R})$:

$$\mathrm{dist}_{\mathbf{X}(\mathcal{R})}(p,q) = \inf\{\ell(\gamma)\colon \gamma \subset \mathbf{X}(\mathcal{R}) \text{ rectifiable arc connecting } p \text{ and } q\}, \quad p,q \in \mathbf{X}(\mathcal{R}).$$

$\mathrm{dist}_{\mathbf{X}(\mathcal{R})}$, $(\mathbf{X}(\mathcal{R}), \mathrm{dist}_{\mathbf{X}(\mathcal{R})}) \equiv$ image distance, image metric space of $\mathbf{X}\colon \mathcal{R} \to \mathbb{C}^2$.

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- $\mathbf{X} \equiv$ image complete if $(\mathbf{X}(\mathcal{R}), \mathrm{dist}_{\mathbf{X}(\mathcal{R})})$ is a complete metric space $\equiv$ if every rectifiable divergent arc in $\mathbf{X}(\mathcal{R})$ has infinite Euclidean length. Image completeness implies completeness, and both notions are equivalent on injective immersions.

Tangent nets

- $\mathcal{D}$ bounded regular strictly convex domain in $\mathbb{C}^2$, $\Delta \subset \mathrm{Fr}\mathcal{D}$ finite set,

$$\Gamma := \bigcup_{p\in\Delta} (p + T_p \,\mathrm{Fr}\mathcal{D}) \subset \mathbb{C}^2 \setminus \mathcal{D}.$$

$\mathcal{T} := \{q \in \mathbb{C}^2 \colon \mathrm{dist}(q, \Gamma) < \epsilon\} \equiv$ tangent net of radius $\epsilon > 0$ for $\mathcal{D}$.

$\mathcal{T}^0 := \Delta \equiv$ 0-skeleton, $\mathcal{T}^1 := \Gamma \equiv$ 1-skeleton.
Given $p \in \mathcal{T}^0$, $\mathcal{T}(p) := \{q \in \mathbb{R}^n \colon \mathrm{dist}(q, p + T_p\mathrm{Fr}\mathcal{D}) < \epsilon\} \equiv$ slab of $\mathcal{T}$ based at p.

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- If $\mathcal{D} \Subset \mathcal{D}'$ and $\epsilon > 0$ is small, then $\ell(\gamma)$ is large (comparatively with $\mathrm{dist}(\mathcal{D}, \mathrm{Fr}\mathcal{D})$) for any arc $\gamma \subset \mathcal{T}$ connecting $\mathrm{Fr}\mathcal{D}$ and $\mathrm{Fr}\mathcal{D}'$.

The tangent net lemma

$$\mathbf{d}(\mathcal{D}, \mathrm{Fr}\mathcal{D}') := \left(\mathrm{dist}(\mathcal{D}, \mathrm{Fr}\mathcal{D}') + \frac{1}{\kappa(\mathcal{D})}\right)\sqrt{\frac{\mathrm{dist}(\mathcal{D}, \mathrm{Fr}\mathcal{D}')}{\mathrm{dist}(\mathcal{D}, \mathrm{Fr}\mathcal{D}') + 2/\kappa(\mathcal{D})}}$$

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- Any convex domain $\mathcal{B}$ of $\mathbb{C}^2$ admits an exhaustion $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ of bounded regular strictly convex domains such that

$$\sum_{n\in\mathbb{N}} \mathbf{d}(\mathcal{D}^n, \mathrm{Fr}\mathcal{D}^{n+1}) = +\infty.$$

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Lemma

Let $\mathcal{D}$ and $\mathcal{D}'$ be bounded regular strictly convex domains in $\mathbb{C}^2$, $\mathcal{D} \Subset \mathcal{D}'$. Let $A \subset \mathrm{Fr}\mathcal{D}$ consisting of a finite collection of smooth closed curves.
Then for any $\epsilon > 0$ there exists a tangent net $\mathcal{T}$ of radius $< \epsilon$ for $\mathcal{D}$ such that

- *$A \subset \mathcal{T}$ and*
- *$\ell(\gamma) > \mathbf{d}(\mathcal{D}, \mathrm{Fr}\mathcal{D}') - \epsilon$ for any Jordan arc $\gamma \subset \mathcal{T}$ connecting $\mathrm{Fr}\mathcal{D}$ and $\mathrm{Fr}\mathcal{D}'$.*

Deforming curves along tangent nets

Lemma

Let $\mathcal{D}$ and $\mathcal{D}'$ be bounded regular strictly convex domains in $\mathbb{C}^2$, $\mathcal{D} \Subset \mathcal{D}'$. Let $\varepsilon > 0$ and let $\mathcal{T}$ be a tangent net of radius ε for $\mathcal{D}$. Let $\mathcal{N}$ be an open connected Riemann surface, let $\mathcal{R} \Subset \mathcal{N}$ be a bordered domain, and let $\mathbf{X} \colon \overline{\mathcal{R}} \to \mathbb{C}^2$ be a holomorphic immersion such that

$$\mathbf{X}(b\overline{\mathcal{R}}) \subset \mathcal{T} \cap \mathrm{Fr}\mathcal{D} \quad \textit{(hence } \mathbf{X}(\overline{\mathcal{R}}) \subset \mathcal{D}\textit{).}$$

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Then, for any $\delta > 0$, there exist a bordered domain $\mathcal{S} \Subset \mathcal{N}$ and a holomorphic immersion $\mathbf{Y}\colon \overline{\mathcal{S}} \to \mathbb{C}^2$ enjoying the following properties:

- *$\mathcal{R} \Subset \mathcal{S}$ and $\overline{\mathcal{S}} \setminus \mathcal{R}$ consists of a finite collection of pairwise disjoint compact annuli.*
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- *$\mathbf{Y}(b\overline{\mathcal{S}}) \subset \mathrm{Fr}\mathcal{D}'$, hence $\mathbf{Y}(\overline{\mathcal{S}}) \subset \overline{\mathcal{D}'}$.*
- *$\mathbf{Y}(\overline{\mathcal{S}}) \subset \mathcal{D} \cup \mathcal{T}$.*

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- *$\mathbf{Y}(\overline{\mathcal{S}}) \subset \mathcal{D} \cup \mathcal{T}$.*

- $T_p\mathrm{Fr}\mathcal{D} \equiv \mathbb{C} \oplus \mathbb{R} = \mathrm{span}_{\mathbb{C}}(u_p) \oplus \mathrm{span}_{\mathbb{R}}(\mathcal{J}(\nu_{\mathcal{D}}(p)))$ has real dimension 3, for all $p \in \mathrm{Fr}\mathcal{D}$.

The desingularization lemma

Lemma

Let $\mathcal{D} \subset \mathbb{C}^2$ *be a strictly convex bounded regular domain. Let* $\mathcal{N}$ *be an open Riemann surface and let* $\mathcal{R}$ *and* $\mathcal{M}$ *be bordered domains in* $\mathcal{N}$*,* $\mathcal{R} \Subset \mathcal{M}$*. Let* $\mathbf{X}\colon \mathcal{N} \to \mathbb{C}^2$ *be a holomorphic immersion satisfying that*

- $\mathbf{X}(b\overline{\mathcal{M}}) \subset \mathrm{Fr}\mathcal{D}$ *(hence* $\mathbf{X}(\overline{\mathcal{R}}) \subset \mathcal{D}$*), and*
- $\mathbf{X}(\overline{\mathcal{R}})$ *contains no double point of* $\mathbf{X}(\overline{\mathcal{M}})$*; in particular,* $\mathbf{X}|_{\overline{\mathcal{R}}}$ *is an embedding.*

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- $\mathbf{X}(b\overline{\mathcal{M}}) \subset \mathrm{Fr}\mathcal{D}$ *(hence $\mathbf{X}(\overline{\mathcal{R}}) \subset \mathcal{D}$), and*
- $\mathbf{X}(\overline{\mathcal{R}})$ *contains no double point of $\mathbf{X}(\overline{\mathcal{M}})$; in particular, $\mathbf{X}|_{\overline{\mathcal{R}}}$ is an embedding.*

Then, for any $\epsilon > 0$ there exist an open Riemann surface $\mathcal{W}$, a bordered domain $\mathcal{S} \Subset \mathcal{W}$, and a holomorphic immersion $\mathbf{F}\colon \mathcal{W} \to \mathbb{C}^2$ such that:

- $\overline{\mathcal{R}} \subset \mathcal{S}$ *(in particular, the closures of $\mathcal{R}$ in $\mathcal{N}$ and $\mathcal{W}$ agree).*
- $\|\mathbf{F} - \mathbf{X}\| < \epsilon$ *on $\overline{\mathcal{R}}$ and the Hausdorff distance*

$$\mathfrak{d}^{\mathrm{H}}\big(\mathbf{X}(\overline{\mathcal{M}} \setminus \mathcal{R}), \mathbf{F}(\overline{\mathcal{S}} \setminus \mathcal{R})\big) < \epsilon.$$

  In particular, $\mathfrak{d}^{\mathrm{H}}\big(\mathbf{X}(\overline{\mathcal{M}}), \mathbf{F}(\overline{\mathcal{S}})\big) < \epsilon$.

- $\mathbf{F}(b\overline{\mathcal{S}}) \subset \mathrm{Fr}\mathcal{D}$.
- $\mathbf{F}|_{\overline{\mathcal{S}}}$ *is an embedding.*

Image complete bounded complex curves

Theorem

Let S be an open orientable smooth surface and let $\mathcal{B} \subset \mathbb{C}^2$ be a convex domain.
Then there exist a complex structure $\mathcal{J}$ on S and an image complete proper holomorphic immersion $(S, \mathcal{J}) \to \mathcal{B}$.

- The proof follows from the same argument but without desingularizing at each step.

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