Geodesic Connectedness in Spacetimes with Lightlike Killing Vector Fields

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Introduction

Euclidean Geometry

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Problem

When may two fixed points be connected by a geodesic?
When is $\mathcal{M}$ geodesically connected?

Geodesic connectedness in Riemannian manifolds

Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$

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Theorem (Hopf–Rinow)

$\mathcal{M}_0$ complete with respect to the distance associated to $\langle\cdot,\cdot\rangle_R$ or, equivalently, $\mathcal{M}_0$ geodesically complete

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$\mathcal{M}_0$ geodesically connected.

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Here, $d(x_1, x_2) = \inf\left\{\int_a^b \sqrt{\langle\gamma',\gamma'\rangle_R}\,ds : \ \gamma \in A_{x_1,x_2}\right\}$
with x_1, $x_2 \in \mathcal{M}_0$ and $\gamma \in A_{x_1,x_2}$ if $\gamma : [a, b] \to \mathcal{M}_0$ is a piecewise smooth curve joining x_1 to x_2.

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Classical variational principle

$\bar{z} : I \to \mathcal{M}$ is a geodesic joining p to q in $\mathcal{M}$

$$\Updownarrow$$

$\bar{z} = \bar{z}(s)$ is a critical point of the action functional

$$f(z) = \int_0^1 g(z(s))[z'(s), z'(s)]\, ds \qquad \text{in } C^1(p, q),$$

with $C^1(p,q) = \{z \in C^1([0,1], \mathcal{M}) : z(0) = p,\ z(1) = q\}$.

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Remark

Without loss of generality, we can take $I = [0, 1]$ as the set of geodesics is invariant by affine reparametrizations.

Abstract tools: existence result

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Definition

J satisfies the Palais–Smale condition on Ω, briefly (PS), if any sequence $(x_k)_k \subset \Omega$ such that

$$(J(x_k))_k \text{ is bounded} \quad \text{and} \quad \lim_{k \to +\infty} J'(x_k) = 0$$

has a subsequence converging in Ω.

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Theorem (Existence)

Let Ω be complete. If J is a functional which satisfies (PS) and is bounded from below, then it attains its infimum.

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Define $H^1(I, \mathcal{M})$ the set of curves $z : I \to \mathcal{M}$ such that for any local chart (U, φ) of $\mathcal{M}$, with $U \cap z(I) \neq \emptyset$, the curve $\varphi \circ z$ belongs to the Sobolev space $H^1(z^{-1}(U), \mathbb{R}^n)$.

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$H^1(I, \mathcal{M})$ is equipped with a structure of infinite dimensional manifold modelled on the Hilbert space $H^1(I, \mathbb{R}^n)$:

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if $z \in H^1(I, \mathcal{M})$, the tangent space to $H^1(I, \mathcal{M})$ at z is

$$T_z H^1(I, \mathcal{M}) \equiv \{\zeta \in H^1(I, T\mathcal{M}) : \pi \circ \zeta = z\},$$

being $T\mathcal{M}$ the tangent bundle of $\mathcal{M}$ and $\pi : T\mathcal{M} \to \mathcal{M}$ the corresponding bundle projection, i.e., $T_z H^1(I, \mathcal{M})$ is the set of the vector fields along z whose components with respect to a local chart are functions of class H^1.

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Fixing p, $q \in \mathcal{M}$, we can consider

$$\Omega^1(p, q; \mathcal{M}) = \{z \in H^1(I, \mathcal{M}) : \; z(0) = p, \, z(1) = q\}$$

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with tangent space

$$T_z\Omega^1(p,q;\mathcal{M}) = \{\zeta \in T_zH^1(I,\mathcal{M}) :\ \zeta(0) = 0 = \zeta(1)\}$$

at $z \in \Omega^1(p,q;\mathcal{M})$.

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is at least of class C^1 on $\Omega^1(p, q; \mathcal{M})$ with

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Theorem ("Weaker" variational principle)

$\bar{z} : I \to \mathcal{M}$ is a geodesic joining p to q in $\mathcal{M}$

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$\bar{z} \in \Omega^1(p, q; \mathcal{M})$ is a critical point of f on $\Omega^1(p, q; \mathcal{M})$.

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Case $\mathcal{M} \equiv \mathbb{R}$

Fixing any t_p, $t_q \in \mathbb{R}$, it is

$$W^1(t_p, t_q) = \{t \in H^1(I, \mathbb{R}) : t(0) = t_p,\ t(1) = t_q\} = H_0^1(I, \mathbb{R}) + j^*,$$

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$$j^* : s \in I \mapsto t_p + s\Delta_t \in \mathbb{R}, \qquad \Delta_t = t_q - t_p.$$

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$j^* : s \in I \mapsto t_p + s\Delta_t \in \mathbb{R}, \qquad \Delta_t = t_q - t_p.$

Whence, $W^1(t_p, t_q)$ is a closed affine submanifold of $H^1(I,\mathbb{R})$ with

$T_t W^1(t_p, t_q) \equiv H^1_0(I,\mathbb{R})$ for all $t \in W^1(t_p, t_q)$.

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By the Nash Embedding Theorem, $\mathcal{M}_0$ (at least C^2) is a submanifold of an Euclidean space $\mathbb{R}^N$ (the embedding is closed in compact regions [Nash '63], complete regions [Müller '09]) and $\langle\cdot,\cdot\rangle_R$ is the restriction to $\mathcal{M}_0$ of the standard Euclidean metric of $\mathbb{R}^N$.

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Hence,

$$H^1(I, \mathcal{M}_0) \equiv \{x \in H^1(I, \mathbb{R}^N) : \ x(I) \subset \mathcal{M}_0\},$$

and, fixing any x_p, $x_q \in \mathcal{M}_0$, we have

$$\Omega^1(x_p, x_q; \mathcal{M}_0) = \{x : I \to \mathcal{M}_0 : \ x \text{ absolutely continuous}, \quad x(0) = x_p,\ x(1) = x_q,\ \int_0^1 \langle x', x'\rangle_R ds < +\infty\}.$$

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$\mathcal{M}_0$ complete $\Longrightarrow$ $H^1(I, \mathcal{M}_0)$, $\Omega^1(x_p, x_q; \mathcal{M}_0)$ complete.

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The energy functional

$$f(x) = \int_0^1 \langle x', x' \rangle_R ds, \quad x \in \Omega^1(x_p, x_q; \mathcal{M}_0)$$

is positive.

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Theorem (Hopf–Rinow)

$\mathcal{M}_0$ *complete as metric space*

$$\Downarrow$$

$\mathcal{M}_0$ *geodesically connected.*

Idea of the proof

Let x_p, $x_q \in \mathcal{M}_0$ be fixed. Denote $\|x'\|^2 = \int_0^1 \langle x', x' \rangle_R ds$.

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Let $x_p,\ x_q \in \mathcal{M}_0$ be fixed. Denote $\|x'\|^2 = \int_0^1 \langle x', x'\rangle_R ds$.

Lemma ("Splitting" Lemma)

Let $\mathcal{M}_0$ be a submanifold of $\mathbb{R}^N$ and $(x_k)_k \subset \Omega^1(x_p, x_q; \mathcal{M}_0)$ a sequence so that $(\|x_k'\|)_k$ is bounded.
Then, $x \in H^1(I, \mathbb{R}^N)$ exists so that, up to subsequences, it is
$x_k \rightharpoonup x$ *weakly in* $H^1(I, \mathbb{R}^N)$, $\quad x_k \to x$ *uniformly in* I.
If $\mathcal{M}_0$ is complete, then $x \in \Omega^1(x_p, x_q; \mathcal{M}_0)$; furthermore, there exist two sequences $(\xi_k)_k$, $(\nu_k)_k \subset H^1(I, \mathbb{R}^N)$ such that
$x_k - x = \xi_k + \nu_k$ *with* $\xi_k \in T_{x_k}\Omega^1(x_p, x_q; \mathcal{M}_0)$ *for all* $k \in \mathbb{N}$,
$\xi_k \rightharpoonup 0$ *weakly and* $\nu_k \to 0$ *strongly in* $H^1(I, \mathbb{R}^N)$.

- V. Benci – D. Fortunato, *Adv. Math.* **105** (1994).

Variational proof

Variational proof of the Hopf–Rinow Theorem:

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Remark

$\mathcal{M}_0$ not contractible in itself $\Longrightarrow$ f has a diverging sequence of critical levels in $\Omega^1(x_p, x_q; \mathcal{M}_0)$.

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From a geometric point of view

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The Hopf–Rinow Theorem cannot be extended to indefinite semi–Riemannian manifolds, in particular to Lorentzian manifolds.

Example (anti–de Sitter spacetime)

$$\mathcal{M} =]-\frac{\pi}{2}, \frac{\pi}{2}[\times\mathbb{R}$$

equipped with the Lorentzian metric

$$\langle\cdot,\cdot\rangle_L = \frac{1}{\cos^2 x}\ \left(dx^2 -\ dt^2\right).$$

$\mathcal{M}$ is geodesically complete but not geodesically connected.

- R. Penrose, *Conf. Board Math. Sci.* **7**, S.I.A.M. (1972).

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No general approach is known.

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No general approach is known.

We select some special semi–Riemannian manifolds (stationary spacetimes, orthogonal splitting spacetimes, Gödel type spacetimes, plane wave type spacetimes, warped product spacetimes, . . .) and develop *ad hoc* techniques.

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- causal if $g(v, v) \leq 0$ and $v \neq 0$;
- spacelike if $g(v, v) > 0$ or $v = 0$.

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- spacelike if $g(v, v) > 0$ or $v = 0$.

Definition

A spacetime is a connected and time–orientable Lorentzian manifold, with a prescribed time–orientation (a continuous choice of a causal cone at each $p \in \mathcal{M}$, which is called the future cone, in opposition to the non–chosen one or past cone).

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Accordingly, causal curves are called either future or past directed depending on the cone of $z'(s)$.

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A C^1 curve $z : I \to \mathcal{M}$ is called timelike, lightlike, spacelike or causal when so it is $z'(s)$ for all $s \in I$.
For causal curves, this definition is extended to piecewise C^1 curves but the two limit tangent vectors on the breaks must belong to the same causal cone.
Accordingly, causal curves are called either future or past directed depending on the cone of $z'(s)$.

Causal character of a geodesic

A (non–constant) geodesic $z : I \to \mathcal{M}$ is:

- timelike if $E_z < 0$;
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with $E_z \equiv \langle z'(s), z'(s)\rangle_L$ for all $s \in I$.

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Other tools of Differential Geometry: Killing vector fields

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Definition

A vector field K is said Killing if the Lie derivative of the metric tensor $\langle\cdot,\cdot\rangle_L$ with respect to K vanishes everywhere, or, equivalently, if the stages of all its local flows are isometries (i.e., $\langle\cdot,\cdot\rangle_L$ is invariant by its flow).

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- if z is only absolutely continuous $\implies$ $\langle z', \nabla^L_s K(z)\rangle_L \equiv 0$ almost everywhere in I.

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- $\langle z'(s), K(z(s))\rangle_L \equiv C_z$ for all $s \in I$.

Globally hyperbolic spacetime

Definition

A spacetime $\mathcal{M}$ is globally hyperbolic if there exists a (smooth) spacelike Cauchy hypersurface $\mathcal{S}$ in $\mathcal{M}$, i.e., a subset which is crossed exactly once by any inextendible timelike curve.

- E. Minguzzi – M. Sánchez. In: *Recent Developments in pseudo-Riemannian Geometry* (D.V. Alekseevsky & H. Baum Eds), EMS Publishing House, 2008.

Stationary Spacetime

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Remark

If one such a timelike Killing vector field K is chosen, then $\mathcal{M}$ is time–oriented.

Standard Stationary Spacetime

Definition

$(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is a standard stationary spacetime if splits globally as $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, with $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ a finite dimensional connected Riemannian manifold, and metric $\langle\cdot,\cdot\rangle_L$ written as

$$\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle_R + 2\langle\delta(x),\cdot\rangle_R dt - \beta(x)dt^2$$

for each $T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, $z = (x,t) \in \mathcal{M}$, where δ and β are a smooth vector field and a smooth strictly positive scalar field on $\mathcal{M}_0$, respectively.
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Fixing $p = (x_p, t_p)$, $q = (x_q, t_q) \in \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, we have:
$\Omega^1(p,q;\mathcal{M}) \equiv \Omega^1(x_p,x_q;\mathcal{M}_0) \times W^1(t_p,t_q)$,
$T_z\Omega^1(p,q;\mathcal{M}) \equiv T_x\Omega^1(x_p,x_q;\mathcal{M}_0) \times H_0^1(I,\mathbb{R})$
in each $z = (x,t) \in \Omega^1(p,q;\mathcal{M})$.

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We can distinguish two different variational approaches:

(a) to transform the indefinite action functional f on $\Omega^1(p, q; \mathcal{M})$ in a new (hopefully bounded from below) functional $\mathcal{J}$ on the Riemannian part $\Omega^1(x_p, x_q; \mathcal{M}_0)$;

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(b) to study directly the strongly indefinite functional f but by making use of suitable (essentially finite–dimensional) "approximating" techniques.

Known results: extrinsic approach

Standard static spacetimes ($\delta \equiv 0$)

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- R. Bartolo, A.M. C., J.L. Flores and M. Sánchez, *Adv. Nonlinear Stud.* **3** (2003)
  If $0 < \beta(x) \leq \lambda d^2(x, \bar{x}) + \mu d^\alpha(x, \bar{x}) + k$.

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- L. Pisani, *Boll. Unione Mat. Ital. A* **7** (1991)
  If some $\alpha < 1$ exists so that $0 < \epsilon \leq \beta(x) \leq \mu d^\alpha(x, \bar{x}) + k$,
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- A.M. C. and A. Salvatore, *J. Geom. Phys.* **44** (2002)
  Multiplicity result in the same case.

Extrinsic approach, method *(a)*

$\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ standard stationary spacetime,
$p = (x_p, t_p)$, $q = (x_q, t_q)$

Proposition (New variational principle)

$z^* = (x^*, t^*) \in \Omega^1(p, q; \mathcal{M})$ is a critical point of the action functional f in $\Omega^1(p, q; \mathcal{M})$ if and only if x^* is a critical point of the functional $\mathcal{J} : \Omega^1(x_p, x_q; \mathcal{M}_0) \to \mathbb{R}$ defined as

$$\mathcal{J}(x) = \int_0^1 \langle x', x' \rangle_R \, ds \; + \; \int_0^1 \frac{\langle \delta(x), x' \rangle_R^2}{\beta(x)} \, ds \\ - \left(\int_0^1 \frac{\langle \delta(x), x' \rangle_R}{\beta(x)} \, ds - \Delta_t \right)^2 \left(\int_0^1 \frac{1}{\beta(x)} \, ds \right)^{-1}$$

and $t^* = \Psi(x^*)$, with $\Delta_t^2 = (t_q - t_p)^2$.
Moreover, $f(z^*) = \mathcal{J}(x^*)$.

Extrinsic approach, method *(a)*

In the previous proposition, it is
$\Psi : \Omega^1(x_p, x_q; \mathcal{M}_0) \to W^1(t_p, t_q)$ defined as

$$\Psi(x)(s) = t_0 + \int_0^s \frac{\langle \delta(x(\sigma)), \dot{x}(\sigma)\rangle}{\beta(x(\sigma))}\, d\sigma - \left(\int_0^1 \frac{\langle \delta(x), \dot{x}\rangle}{\beta(x)}\, ds - \Delta_t\right) \int_0^s \frac{1}{\beta(x(\sigma))}\, d\sigma \left(\int_0^1 \frac{1}{\beta(x)}\, ds\right)^{-1}.$$

- F. Giannoni and A. Masiello, *J. Funct. Anal.* **101** (1991).

Extrinsic approach, method *(a)*

Theorem

Let $\mathcal{M} = \mathbb{R} \times \mathcal{M}_0$ *be a standard stationary spacetime.*
If $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ *is complete and* $\bar{x} \in \mathcal{M}_0$ *exists such that*

$$0 < \beta(x) \leq \lambda d^2(x,\bar{x}) + \mu d^{\alpha}(x,\bar{x}) + k,$$

$$\sqrt{\langle\delta(x),\delta(x)\rangle_R} \leq \lambda_1\, d(x,\bar{x}) + \mu_1 d^{\alpha_1}(x,\bar{x}) + k_1,$$

for all $x \in \mathcal{M}_0$ *and for suitable* $\lambda, \lambda_1 \geq 0$, $\mu, \mu_1, k, k_1 \in \mathbb{R}$, $\alpha, \alpha_1 \in [0,1)$.
Then, $\mathcal{M}$ *is geodesically connected.*

- R. Bartolo, A.M. C. and J.L. Flores, *J. Geom. Phys. 56* (2006)

Idea of the proof

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Remark

$\mathcal{M}_0$ non–contractible in itself $\Longrightarrow$ any two points can be joined by a sequence of (spacelike) geodesics $(z_k)_k$ with diverging lengths.

Known results: intrinsic approach

- F. Giannoni and P. Piccione, *Comm. Anal. Geom.* **7** (1999).

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  Applying the previous result to the standard stationary case:

  $0 < \epsilon \leq \beta(x) \leq M$,
  $|\delta(x)| \leq \mu d^{\alpha}(x, \bar{x}) + k$,
  for all $x \in \mathcal{M}_0$ $(\alpha < 1)$.

Remarks

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Every stationary spacetime is locally a standard stationary one with $K = \partial_t$ as timelike Killing vector field.

Theorem

A globally hyperbolic stationary spacetime is a standard stationary one, if one of its timelike Killing vector fields is complete.

- A.M. C., J.L. Flores and M. Sánchez, *Adv. Math.* **218** (2008)

A more general case

A more general result can be stated:

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Theorem (Part 1)

Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ be a globally hyperbolic spacetime admitting a complete causal Killing vector field K. Then, there exist a Riemannian manifold $(\mathcal{S}, \langle\cdot,\cdot\rangle)$, a differentiable vector field δ on $\mathcal{S}$ and a differentiable non–negative function β on $\mathcal{S}$ such that $\mathcal{M} = \mathcal{S} \times \mathbb{R}$ and

$$\langle\zeta,\zeta'\rangle_L = \langle\xi,\xi'\rangle + \langle\delta(x),\xi\rangle\tau' + \langle\delta(x),\xi'\rangle\tau - \beta(x)\tau\tau',$$

for all $z = (x,t) \in \mathcal{M}$ and $\zeta = (\xi,\tau), \zeta' = (\xi',\tau') \in T_z\mathcal{M} = T_x\mathcal{S} \times \mathbb{R}$.

- R. Bartolo, A.M. C. and J.L. Flores, *Rev. Mat. Iberoam.* (?)

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- K *timelike* $\Longrightarrow$ β *is non–vanishing, i.e.,* $\beta(x) > 0$ *for all* $x \in \mathcal{S}$*;*
- K *lightlike* $\Longrightarrow$ $\beta \equiv 0$, δ *is non–vanishing and the metric on* $\mathcal{M}$ *becomes*

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau,$$

for all $z = (x, t) \in \mathcal{M}$ *and* $\zeta = (\xi, \tau), \zeta' = (\xi', \tau') \in T_z\mathcal{M} = T_x\mathcal{S} \times \mathbb{R}$*.*

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If $z \in C^1_K(p,q)$ is a critical point of f restricted to $C^1_K(p,q)$, then z is a geodesic connecting p to q.

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Define $\Omega^1_K(p,q) = \{z \in \Omega^1(p,q;\mathcal{M}) : \exists C_z \in \mathbb{R}$ such that $\langle z', K(z)\rangle_L = C_z$ a.e. on $I\}$.

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Intrinsic approach

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Definition

Fixed $c \in \mathbb{R}$ the set $\Omega^1_K(p,q)$ is c–precompact for f if every sequence $(z_n)_n \subset \Omega^1_K(p,q)$ with $f(z_n) \le c$ has a subsequence which converges weakly in $\Omega^1(p,q;\mathcal{M})$ (hence, uniformly in $\mathcal{M}$). Furthermore, the restriction of f to $\Omega^1_K(p,q)$ is pseudo–coercive if $\Omega^1_K(p,q)$ is c–precompact for all $c \ge \inf f(\Omega^1_K(p,q))$.

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If $\Omega^1_K(p,q)$ is not empty and there exists $c > \inf f(\Omega^1_K(p,q))$ such that $\Omega^1_K(p,q)$ is c–precompact, then there exists at least one geodesic joining p to q in $\mathcal{M}$

- F. Giannoni and P. Piccione, *Comm. Anal. Geom.* **7** (1999)

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Remark

The main limitation of Giannoni and Piccione's results is that pseudo–coercivity condition is analytical and very technical. Furthermore, in general, the assumption $\Omega^1_K(p,q)$ non–empty must be imposed.

Intrinsic approach

In order to overcome the limitation of Giannoni and Piccione's result, the idea is to introduce purely geometric assumptions on $\mathcal{M}$.

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Theorem

Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ be a stationary spacetime with a complete timelike Killing vector field K. If $\mathcal{M}$ is globally hyperbolic with a complete (smooth, spacelike) Cauchy hypersurface $\mathcal{S}$, then it is geodesically connected.

- A.M. C., J.L. Flores and M. Sánchez, *Adv. Math.* **218** (2008)

Hint of the proof

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If the timelike Killing vector field K is complete, then it is $\Omega^1_K(p,q) \neq \emptyset$ for each $p, q \in \mathcal{M}$.

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If the timelike Killing vector field K is complete and $\mathcal{M}$ is globally hyperbolic with a complete Cauchy hypersurface, then the restriction of f to $C^1_K(p,q)$ is pseudo–coercive, for any p, $q \in \mathcal{M}$.

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Hence, the Giannoni - Piccione Theorem applies.

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Lightlike Killing case

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In general, the answer is: NO.

Counterexample

$\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}$ equipped with the Lorentzian metric

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \, \tau' + \langle \delta(x), \xi' \rangle \, \tau,$$

for all $\zeta = (\xi, \tau), \zeta' = (\xi', \tau') \in \mathbb{R}^4$, where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^3$ and
$\delta : x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \delta(x_1) \in \mathbb{R}^3$ satisfies

$$\delta(x_1) = \begin{cases} (-\cos^3 x_1, 0, 0) & \text{if } x_1 < \pi \\ (1, 0, 0) & \text{if } x_1 \geq \pi. \end{cases}$$

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Theorem

Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ be a globally hyperbolic spacetime endowed with a complete lightlike Killing vector field K and a complete (smooth, spacelike) Cauchy hypersurface $\mathcal{S}$. Let $p, q \in \mathcal{M}$. Then:

p and q are geodesically connected in $\mathcal{M}$

$$\Updownarrow$$

p and q can be connected by a C^1 curve φ on $\mathcal{M}$ such that $\langle\dot\varphi, K(\varphi)\rangle_L$ has constant sign or is identically equal to 0.

- R. Bartolo, A.M. C. and J.L. Flores, *Rev. Mat. Iberoam.* (?)

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- We introduce a sequence of standard stationary spacetimes $(\mathcal{M}_n, \langle \cdot, \cdot \rangle_n)$ such that: $\mathcal{M}_n = \mathcal{M}$ and $\langle \cdot, \cdot \rangle_n = \langle \cdot, \cdot \rangle_L - \frac{1}{n} dt^2$

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- Taking $p = (x_p, t_p)$, $q = (x_q, t_q) \in \mathcal{M}$ with $\Delta_t = t_q - t_p \geq 0$, there exists $n_0 \in \mathbb{N}$ such that p and q are connected by a geodesic $\gamma_n = (x_n, t_n)$ in $(\mathcal{M}_n, \langle \cdot, \cdot \rangle_n)$ for every $n \geq n_0$

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- If a C^1 curve φ exists such that $\langle \dot{\varphi}, K(\varphi) \rangle_L$ has constant sign or is identically equal to 0, then γ exists such that, up to subsequences, $\gamma_n \to \gamma$ strongly on $\Omega(x_p, x_q; \mathcal{S}) \times W(t_p, t_q)$

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- γ is a geodesic joining p to q in $\mathcal{M}$.

Applications: Avez–Seifert result

An alternative proof of the classical Avez–Seifert result:

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An alternative proof of the classical Avez–Seifert result:

Proposition

Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ be a globally hyperbolic spacetime endowed with a complete lightlike Killing vector field K and a complete Cauchy hypersurface $\mathcal{S}$. Then, two points of $\mathcal{M}$ can be connected by a causal geodesic if and only if they are causally related.

Applications: Generalized Plane Waves

Definition

A Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is called *generalized plane wave*, briefly *GPW*, if there exists a (connected) finite dimensional Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ and

$$\langle\cdot,\cdot\rangle_L \;=\; \langle\cdot,\cdot\rangle_R + 2dudv + \mathcal{H}(x,u)du^2,$$

where $x \in \mathcal{M}_0$, the variables (u, v) are the natural coordinates of $\mathbb{R}^2$ and the smooth function $\mathcal{H} : \mathcal{M}_0 \times \mathbb{R} \to \mathbb{R}$ is not identically zero.

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A GPW becomes a gravitational wave if $\mathcal{M}_0 = \mathbb{R}^2$ is equipped with the classical Euclidean metric and
$\mathcal{H}(x,u) \;=\; g_1(u)(x_1^2 - x_2^2) + 2g_2(u)x_1x_2,\ x = (x_1,x_2) \in \mathbb{R}^2$,
with g_1, g_2 smooth real functions such that $g_1^2 + g_2^2 \neq 0$.

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About geodesic connectedness and global hyperbolicity of GPWs:

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Theorem

If the Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ is complete with respect to its canonical distance and $\mathcal{H}$ behaves subquadratically at spatial infinity, i.e., there exist $\bar{x} \in \mathcal{M}_0$ and (positive) continuous functions $R_1(u)$, $R_2(u)$, $p(u)$, with $p(u) < 2$, such that

$$-\mathcal{H}(x,u) \leq R_1(u) d^{p(u)}(x,\bar{x}) + R_2(u) \quad \text{for all } (x,u) \in \mathcal{M}_0 \times \mathbb{R},$$

then the spacetime is geodesically connected and globally hyperbolic.

- A.M. C., J.L. Flores, M. Sánchez, *Gen. Relativity Gravitation* **35** (2003)
- J.L. Flores, M. Sánchez, *Class. Quant. Grav.* **20** (2003)

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If $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is a GPW, then $K = \partial_v$ is a complete lightlike Killing vector field on $\mathcal{M}$.

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If $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is a GPW, then $K = \partial_v$ is a complete lightlike Killing vector field on $\mathcal{M}$.
Moreover, taking any $p = (x_p, u_p, v_p), q = (x_q, u_q, v_q) \in \mathcal{M}$ and any curve $x = x(s)$ in $\mathcal{M}_0$ connecting x_p to x_q, if we denote $\Delta_u = u_q - u_p$ and $\Delta_v = v_q - v_p$, we have that the curve $\varphi(s) = (x(s), \Delta_u\, s, \Delta_v\, s)$ connects p to q, and the scalar product

$$\langle\dot\varphi, K(\varphi)\rangle_L = \dot u = \Delta_u$$

is constant.

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Moreover, taking any $p = (x_p, u_p, v_p), q = (x_q, u_q, v_q) \in \mathcal{M}$ and any curve $x = x(s)$ in $\mathcal{M}_0$ connecting x_p to x_q, if we denote $\Delta_u = u_q - u_p$ and $\Delta_v = v_q - v_p$, we have that the curve $\varphi(s) = (x(s), \Delta_u\, s, \Delta_v\, s)$ connects p to q, and the scalar product

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Theorem

Any globally hyperbolic GPW with a complete Cauchy hypersurface is geodesically connected.