Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds

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- u is a *stable* solution of (1).

Solutions of (1) are critical points of the energy functional E given by

$$E(w) = \frac{1}{2}\int |\nabla w|^2 \mathrm{d}x - \int F(w)\mathrm{d}x, \qquad \text{where} \quad F(t) = \int_0^t f(s)\mathrm{d}s, \quad (2)$$

with respect to compactly supported variations.

Stable solution

Definition

The function u is said to be a **stable solution** of (1) if

$$\int_M |\nabla\phi|^2 \mathrm{d}x - \int_M f'(u)\phi^2 \mathrm{d}x \geq 0 \qquad \text{for every } \phi \in C_c^\infty(M) \qquad (3)$$

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or,

if the the Jacobi operator of E at u,

$$J\phi = -\Delta\phi - f'(u)\phi \qquad \forall\, \phi \in C_c^\infty(M), \tag{4}$$

is non-negative on $C_c^\infty(M)$.

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Let $(M, \langle\, , \rangle)$ be a complete, non-compact Riemannian manifold without boundary, satisfying $\mathrm{Ric} \geq 0$. Suppose that $u \in C^3(M)$ be a non-constant, stable solution of $-\Delta u = f(u)$, for $f \in C^1(\mathbb{R})$. If either

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(i) *M is parabolic and $\nabla u \in L^\infty(M)$, or*

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$$\int_{B_R} |\nabla u|^2 \mathrm{d}x = o(R^2 \log R) \qquad as\ R \to +\infty. \tag{5}$$

Then,

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- $M = N \times \mathbb{R}$ *with the product metric* $\langle\, , \rangle = \langle\, , \rangle_N + \mathrm{d}t^2$, *for some complete, totally geodesic, parabolic hypersurface* N. *In particular,* $\mathrm{Ric}^N \geq 0$ *if* $m \geq 3$, *and* $M = \mathbb{R}^2$ *or* $\mathbb{S}^1 \times \mathbb{R}$, *with their flat metric, if* $m = 2$;

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Moreover, if (ii) is met,

$$\mathrm{vol}(B_R^N) = o(R^2 \log R) \qquad \text{as } R \to +\infty. \tag{6}$$

$$\int_{-R}^{R} |y'(t)|^2 \mathrm{d}t = o\left(\frac{R^2 \log R}{\mathrm{vol}(B_R^N)}\right) \qquad \text{as } R \to +\infty. \tag{7}$$

Sketch of proof

- u **stable** $\Longrightarrow$ the existence of a smooth function w such that :

$$\begin{cases} \Delta w + f'(u)w = 0 & \text{on} \quad \mathrm{M} \\ w > 0 & \text{on} \quad \mathrm{M} \end{cases} \tag{8}$$

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- Integrate $\Delta w + f'(u)w = 0$ against the test function ψ^2, $\psi \in \mathrm{Lip}_c(M)$, to deduce :

$$\int w^2 \left| \nabla \left(\frac{\psi}{w} \right) \right|^2 \mathrm{dx} \quad \leq \quad \int |\nabla \psi|^2 \mathrm{dx} - \int f'(u)\psi^2 \mathrm{dx} \tag{9}$$

- Since u solves $-\Delta u = f(u)$ on M, the Böchner formula gives :

$$\frac{1}{2}\Delta|\nabla u|^2 = -f'(u)|\nabla u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + |\nabla \mathrm{d} u|^2. \qquad (10)$$

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Integrating the latter against the test function ϕ^2, $\phi \in \mathrm{Lip}_c(M)$, we get :

$$\int \left[|\nabla \mathrm{d} u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\right] \phi^2 \mathrm{d}x$$

$$= \int f'(u)|\nabla u|^2\phi^2 \mathrm{d}x - \int \phi\langle\nabla\phi, \nabla|\nabla u|^2\rangle \mathrm{d}x. \tag{11}$$

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$$\int \left[|\nabla \mathrm{d} u|^2 - \left|\nabla|\nabla u|\right|^2 + \mathrm{Ric}(\nabla u, \nabla u)\right] \phi^2 \mathrm{d}x$$
$$\leq \int |\nabla \phi|^2 |\nabla u|^2 \mathrm{d}x - \int w^2 \left|\nabla\left(\frac{\phi|\nabla u|}{w}\right)\right|^2 \mathrm{d}x. \tag{12}$$

$$\int \left[|\nabla \mathrm{d} u|^2 - \left|\nabla |\nabla u|\right|^2 + \mathrm{Ric}(\nabla u, \nabla u) \right] \phi^2 \mathrm{d}x$$

$$+(1-\delta) \int \phi^2 w^2 \left| \nabla \left(\frac{|\nabla u|}{w} \right) \right|^2 \mathrm{d}x \le \frac{1}{\delta} \int |\nabla \phi|^2 |\nabla u|^2 \mathrm{d}x \qquad (13)$$

for some $0 < \delta < 1$.

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In case (*ii*), we apply a logarithmic cutoff argument. For fixed $R > 0$, choose the following radial $\phi(x) = \phi_R(r(x))$:

$$\phi_R(r) = \begin{cases} 1 & \text{if } r \le \sqrt{R}, \\ 2 - 2\dfrac{\log r}{\log R} & \text{if } r \in [\sqrt{R}, R], \\ 0 & \text{if } r \ge R. \end{cases} \qquad (14)$$

$$|\nabla u| = cw, \quad \text{for some } c \geq 0, \tag{15}$$

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Let u be a C^2 function on M, and let $p \in M$ be a point such that $\nabla u(p) \neq 0$. Then, denoting with II the second fundamental form of the level set $\Sigma = \{u = u(p)\}$ in a neighbourhood of p, it holds

$$|\nabla \mathrm{d}u|^2 - \big|\nabla |\nabla u|\big|^2 = |\nabla u|^2 |II|^2 + \big|\nabla_T |\nabla u|\big|^2,$$

where ∇_T is the tangential gradient on the level set Σ.

$$|\nabla u| > 0, \tag{17}$$

$$|\nabla \mathrm{d}u|^2 = \big|\nabla|\nabla u|\big|^2, \quad \mathrm{Ric}(\nabla u, \nabla u) = 0, \tag{18}$$

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The flow Φ_t of the unit vector field $\nu = \nabla u/|\nabla u|$ is well defined on M and, since ∇u is constant on the level sets of u, it moves level sets of u onto level sets of u. Therefore, having chosen a level set N, the map $\Phi : N \times \mathbb{R} \to M$ is a diffeomorphism.

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• In a local Darboux frame $\{e_j, \nu\}$ for the level surface N,

$$\begin{aligned} 0 &= |II|^2 \Longrightarrow \nabla \mathrm{d}u(e_i, e_j) = 0 \\ 0 &= \langle \nabla|\nabla u|, e_j\rangle = \nabla \mathrm{d}u(\nu, e_j), \end{aligned} \tag{20}$$

so the unique nonzero component of $\nabla \mathrm{d}u$ is that corresponding to the pair (ν, ν). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(u \circ \gamma) = \langle \nabla u, \nu\rangle = |\nabla u| \circ \gamma > 0,$$

where γ is any integral curve of ν,

$$\begin{aligned} -f(u \circ \gamma) &= \Delta u(\gamma) = \nabla \mathrm{d}u(\nu, \nu)(\gamma) = \langle \nabla|\nabla u|, \nu\rangle(\gamma) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u| \circ \gamma) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(u \circ \gamma), \end{aligned}$$

hence $y = u \circ \gamma$ solves the ODE $-y'' = f(y)$.

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Let $u \in C^2(\mathbb{R}^m, [-1,1])$ *satisfy*

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The original conjecture has been proven in dimensions $m = 2, 3$ and it is still open, in its full generality, for $4 \leq m \leq 8$.

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Let X be a Killing vector field on $(M^m, \langle\, ,\, \rangle)$ and let $u \in C^3(M)$ be a solution of $-\Delta u = f(u)$, for some $f \in C^1(\mathbb{R})$. Then, the function $w = \langle \nabla u, X \rangle$ solves

$$\Delta w + f'(u) w = 0 \qquad \text{on } M. \tag{22}$$

An extended version of De Giorgi's conjecture

Theorem

Let $(M, \langle\, , \rangle)$ be a complete non-compact Riemannian manifold without boundary with $\mathrm{Ric} \geq 0$ and let X be a Killing field on M. Suppose that $u \in C^3(M)$ is a solution of

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then,

$M = N \times \mathbb{R}$ *with the product metric* $\langle \, , \, \rangle = \langle \, , \, \rangle_N + \mathrm{d}t^2$, *for some complete, totally geodesic, parabolic submanifold* N. *In particular,* $\mathrm{Ric}^N \geq 0$ *if* $m \geq 3$, *while, if* $m = 2$, $M = \mathbb{R}^2$ *or* $\mathbb{S}^1 \times \mathbb{R}$ *with their flat metric.*

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Furthemore, u depends only on t and writing $u = y(t)$ it holds that

$$-y'' = f(y), \qquad y' > 0.$$

Corollary

Let $(M, \langle\, ,\, \rangle)$ be a complete non-compact surface without boundary, with Gaussian curvature $K \geq 0$ and let X be a Killing field on M. Suppose that $u \in C^3(M)$ is a solution of

$$\begin{cases} -\Delta u = f(u) & \text{on } M \\ \langle \nabla u, X \rangle > 0 & \text{on } M \\ \nabla u \in L^\infty(M) & \end{cases}$$

with $f \in C^1(\mathbb{R})$.
Then, M is the Riemannian product $\mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}$, with flat metric, u depends only on t and, writing $u = y(t)$, it holds

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Serrin's problem

Motivated by a problem arising in fluid mechanics, *J. Serrin* considered the overdetermined boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0, \quad \frac{\partial u}{\partial \nu} = const. & \text{on } \partial\Omega \end{cases} \tag{23}$$

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and proved the following celebrated result

Theorem (J. Serrin, 1971)

Let $\Omega \subset \mathbb{R}^N$ be a connected bounded open set of class C^2 and let $f \in C^1$. If the overdetermined boundary value problem (23) admits a $C^2(\overline{\Omega})$, then Ω must be a ball and u is radially symmetric about its center.

Serrin's problem in unbounded domains

In 1997, *H. Berestycki, L. Caffarelli* and *L. Nirenberg*, motivated by questions on the regularity of some one-phase free boundary problems, are led to the study of semilinear problems of *bistable type* in *globally Lipschitz unbounded* domains.

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- They considered the case of a *smooth, globally Lipschitz epigraph*, i.e. a domain Ω of the form :

$$\Omega := \{ (x', x_N) \in \mathbb{R}^N \, : \, \varphi(x') < x_N \},$$

where $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ is a globally Lipschitz smooth function.

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the *Allen-Cahn equation.*

Theorem (F. - Valdinoci, 2010)

Let $f \in C^1$ be of bistable type and Ω a globally lipschitz smooth epigraph of $\mathbb{R}^N$, with $\underline{N = 2, 3}$. If the overdetermined boundary value problem

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admits a bounded $C^2(\overline{\Omega})$-solution, then up to isometry *Ω is the half-space $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, +\infty)$ and u is one-dimensional and monotone (that is $u = u(x_N)$ and $\frac{\partial u}{\partial x_N} > 0$).*

Overdetermined bvp in a Riemannian setting

Theorem

Let $(M, \langle\, ,\, \rangle)$ be a complete, non-compact Riemannian manifold without boundary, satisfying $\mathrm{Ric} \geq 0$ and let X be a Killing field on M. Let $\Omega \subseteq M$ be an open and connected set with C^3 boundary. Suppose that $u \in C^3(\overline{\Omega})$ is a non-constant solution of the overdetermined problem

$$\begin{cases} -\Delta u = f(u) & \text{on } \Omega \\ u = \text{ constant} & \text{on } \partial\Omega \\ \partial_\nu u = \text{ constant} \neq 0 & \text{on } \partial\Omega \end{cases} \tag{25}$$

such that $\langle \nabla u, X \rangle$ is either positive or negative on Ω.

Theorem

Then, if either

(i) *M is parabolic and $\nabla u \in L^\infty(\Omega)$, or*

(ii) *the function $|\nabla u|$ satisfies*

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the following properties hold true:

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- *$\Omega = \partial\Omega \times \mathbb{R}^+$ with the product metric $\langle \, , \, \rangle = \langle \, , \, \rangle_{\partial\Omega} + \mathrm{d}t^2$, $\partial\Omega$ is totally geodesic in M and satisfies $\mathrm{Ric}_{\partial\Omega} \geq 0$,*
- *the function u depends only on t, it has no critical points, and writing $u = y(t)$ it holds $-y'' = f(y)$,*
- *if (ii) is met, $\partial\Omega$ satisfies $\mathrm{vol}(B_R^{\partial\Omega}) = o(R^2 \log R)$ as $R \to +\infty$.*

$$\int_\Omega \left[|\nabla \mathrm{d} u|^2 - \big|\nabla|\nabla u|\big|^2 + \mathrm{Ric}(\nabla u, \nabla u) \right] \phi^2 \mathrm{d}x$$
$$\leq \int_\Omega |\nabla \phi|^2 |\nabla u|^2 \mathrm{d}x - \int_\Omega w^2 \left| \nabla \left(\frac{\phi |\nabla u|}{w} \right) \right|^2 \mathrm{d}x. \tag{26}$$

For every $\phi \in \underline{\mathrm{Lip}_c(M)}$ and <u>not only</u> in $\mathrm{Lip}_c(\Omega)$.

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- Here we crucially use the *overdetermined conditions*

$$u = 0, \qquad \frac{\partial u}{\partial \nu} = const. \qquad on \qquad \partial\Omega$$

Conjecture (Berestycki, Caffarelli, Nirenberg, 1997)

Assuming that $\Omega \subset \mathbb{R}^N, N \geq 2$, *is a smooth domain with* Ω^c *connected and that there exists a* bounded *smooth solution of the overdetermined boundary value problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0, \quad \frac{\partial u}{\partial \nu} = const. & \text{on } \partial\Omega \end{cases} \tag{27}$$

for some Lipschitz function f, then Ω *is either a half-space, a ball, a circular-cylinder-type domain:* $\mathbb{R}^j \times B$, *with B a ball in* $\mathbb{R}^{N-j}$ *or the complement of one these regions.*

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Ω is a periodic perturbation of the circular cylinder $B \times \mathbb{R} \subset \mathbb{R}^N$, where B is the unit ball of $\mathbb{R}^{N-1}$, also $N \geq 3$ and hence Ω^c is connected.

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This leaves **open** the conjecture (BCN) for $N = 2$.

- *(A. Ros - P. Sicbaldi (2012))*

In dimension $N = 2$, the conjecture (BCN) is true in the following two cases:

(i) when Ω is contained in a half-plane and $|\nabla u|$ is bounded, or

(ii) when there exists a positive constant λ such that $f(t) \geq \lambda t$, $\forall\, t > 0$.

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- *M. Traizet* (2013) has considered the case of harmonic functions in the plane (i.e. $f \equiv 0$).
He gives a classification for unbounded domains $\Omega \subset \mathbb{R}^2$ of *finite connectivity* (meaning that $\partial\Omega$ has a finite number of components).

Find "reasonable" conditions on the open domain $\Omega \subset \mathbb{R}^N$ and on the function f leading to a classification (of Ω and/or u).

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Same question in a Riemannian setting.

Thank you for your attention!