The Willmore and other L^2 curvature functionals in Riemannian manifolds

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Introduction

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- Σ closed (compact, $\partial\Sigma = \emptyset$) 2-d surface
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- $A_{ij} = - < \nabla_{\partial_{x^i}} N, \partial_{x^j} >$ II fundamental form of $f(\Sigma)$
- $H = A_{ij}\mathring{g}^{ij} = k_1 + k_2 =$ mean curvature
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Question

Which are the best immersions f?

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How it is possible to relax the definitions in order to get existence?

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Remark

(M, g) and Σ are fixed at the beginning, minimize in the immersion f

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Definition

In case $p = 2$,

$$W(f) := \frac{1}{4}\int_{f(\Sigma)} |H|^2 \quad \textit{Willmore functional}$$

$$W_{cnf}(f) := \frac{1}{2}\int_{f(\Sigma)} |A^\circ|^2 \quad \textit{Conformal Willmore functional}$$

$$E(f) := \frac{1}{2}\int_{f(\Sigma)} |A|^2 \quad \textit{Energy functional}$$

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Remark: if $(M, g) = (\mathbb{R}^3, eucl)$ then by Gauss Bonnet Theorem

$$W(f) = W_{cnf}(f) + 2\pi\chi_E(\Sigma) = \frac{1}{2}E(f) + \pi\chi_E(\Sigma)$$

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Theorem (Wiener)

$W_{cnf} = \frac{1}{2}\int |A^\circ|^2$ is conformally invariant, i.e.

$$\forall u \in C^\infty(M) \text{ called } g[u] := e^{2u}g \Rightarrow W_{cnf}(f)[u] = W_{cnf}(f)$$

where $W_{cnf}(f)[u]$ is the conformal Willmore functional evaluated on $f(\Sigma)$ immersed in $(M, g[u])$.

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Remark

W is conformal invariant in $\mathbb{R}^3$ but not in a general manifold $\Rightarrow$ W_{cnf} is the "correct" Willmore functional from a conformal point of view.

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$$Mass(\Sigma) := \sqrt{\frac{Area(\Sigma)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma |H|^2\right)$$

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- Nonlinear elasticity theory, as Γ-limit of energy functionals (Friesecke-James-Müller)

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GOAL: Minimize or more generally find critical points of W, W_{cnf}, E and related functionals

$\rightarrow$ prove existence of "generalized special immersions".

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  $\forall \Sigma, \forall f : \Sigma \hookrightarrow \mathbb{R}^3 \Rightarrow W(f) \geq 4\pi$ and $W(f) = 4\pi \Leftrightarrow f(\Sigma) = S_p^\rho$

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TODAY: give results, i.e. existence or non-existence of minimizers or critical points, in non constantly curved manifolds

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Existence for W in $(\mathbb{R}^3, g_\epsilon)$

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Theorem [M.(Math. Zeit. '10)]
Assume
- $\exists \bar{p} \in \mathbb{R}^3$ such that $R_1(\bar{p}) \neq 0$,
- Said $\|h(p)\| := \sup_{|v|=1} |h_p(v, v)|$
  $i)$ $\lim_{|p|\to\infty} \|h(p)\| = 0$.
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Then, for ϵ small enough, there exists a perturbed standard sphere $S^{\rho_\epsilon}_{p_\epsilon}(w_\epsilon(p_\epsilon, \rho_\epsilon))$ (where $w_\epsilon(p_\epsilon, \rho_\epsilon) \in C^{4,\alpha}(S^2)$) which is a Willmore immersion of $\mathbb{S}^2$ in $(\mathbb{R}^3, g_\epsilon)$

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Lemma[M. (Math. Zeit. '10)]:Let (M, g) be a general ambient manifold with scalar curvature R, then the following expansion of W on small geodesic spheres hold:

$$W(S_{p,\rho}) = 4\pi - \frac{2\pi}{3} R(p)\rho^2 + O_p(\rho^3)$$

Remark + non Existence for W

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$$R(\bar{p}) \neq 0.$$

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REMARK: different behavior from flat case where all the spheres are critical points

Results for $W_{cnf} = \frac{1}{2}\int |A^{\circ}|^2$ in $(\mathbb{R}^3, g_{\epsilon})$

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NOTATION: traceless Ricci tensor of a Riemannian manifold
$S_{\mu\nu} := Ric_{\mu\nu} - \frac{1}{3} R\, g_{\mu\nu}$
in $(\mathbb{R}^3, g_\epsilon)$ one has $\|S_p\|^2 = \epsilon^2 \tilde{s}_p + o(\epsilon^2)$

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Theorem (M. (J.G.A. '11))

Let $h_{\mu\nu} \in C_0^\infty(\mathbb{R}^3)$ and let c be such that

$$c := \sup\{\|h_{\mu\nu}\|_{H^1(\pi)} : \pi \textit{ is an affine plane in } \mathbb{R}^3,\ \mu, \nu = 1, 2, 3\}.$$

Then there exists a constant $A_c > 0$ depending on c with the following property: if there exists a point $\bar{p}$ such that

$$\tilde{s}_{\bar{p}} > A_c$$

then, for ϵ small enough, there exists a perturbed standard sphere $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon)$ which is a critical point of the conformal Willmore functional W_{cnf} converging to a standard sphere as $\epsilon \to 0$.

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Theorem (M. (J.G.A. '11))

Let (M,g) be a Riemannian manifold. Assume that the traceless Ricci tensor of M at the point $\bar{p}$ is not null:

$$\|S_{\bar{p}}\| \neq 0.$$

Then there exist $\rho_0 > 0$ and $r > 0$ such that for radius $\rho < \rho_0$ and perturbation $w \in C^{4,\alpha}(S^2)$ with $\|w\|_{C^{4,\alpha}(\mathbb{S}^2)} < r$, the surfaces $S_{\bar{p},\rho}(w)$ are not critical points of W_{cnf}.

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Remark:- The condition $\|S_p\| \neq 0$ is generic.
- If (M, g) has not constant sectional curvature then there exists at least one point $\bar{p}$ such that $\|S_{\bar{p}}\| \neq 0$.

Minimization of $E = \frac{1}{2}\int |A|^2$ in compact manifolds: global setting

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Theorem (Kuwert- M.- Schygulla '11)

Let (M, g) be a compact 3-dimensional Riemannian manifold with strictly positive sectional curvature $\bar{K} > 0$.

Then there exists a smooth immersion $f : \mathbb{S}^2 \hookrightarrow M$ such that

$$E(f) = \inf\{E(h) | h : \mathbb{S}^2 \hookrightarrow (M, g) \text{ is a } C^\infty \text{ immersion in } (M, g)\}.$$

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REMARK- By compactness there exists a $\lambda > 0$ such that

$$\bar{K} \geq \lambda > 0. \tag{1}$$

- the theorem is non trivial in the sense that there are examples of compact 3-manifolds with $\bar{K} > 0$ which do not contain totally geodesic immersions; for instance Berger Spheres [Souam-Toubiana (Comm. Math. Helv. '09)]

Sketch of proof-1:Framework

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- enlarge the space for proving compactness of the minimizing sequence
- prove that the minimizing sequence does not degenerate

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Enlarged space: take a minimizing sequence of immersions
$f_k : \mathbb{S}^2 \hookrightarrow M$
associate the Radon measures
$\mu_k : \mu_k(B) := Area_{(f_k^* g)}(f_k^{-1}(B))'' = Area(B \cap f_k(\mathbb{S}^2))''$ for every $B \subset M$ Borel set

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Lemma Let (M, g) be a closed 3-dimensional manifold with positive sectional curvature $\bar{K}$: $\exists\lambda$ such that $\bar{K} > \lambda^2 > 0$. Then, for every smooth immersion $f : \mathbb{S}^2 \hookrightarrow (M, g)$, the following area estimate holds:

$$|f(\mathbb{S}^2)|_g \leq \frac{1}{\lambda^2}\Big(4\pi + E(f)\Big) \tag{2}$$

where $|f(\mathbb{S}^2)|_g := \int_{\mathbb{S}^2} d\mu_g$ is the area of $\mathbb{S}^2$ equipped with the pull back metric f^*g given by the immersion.

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Idea of Proof: play with the Gauss equation

$$\bar{K}(T_x f) = K_G - k_1 k_2 = K_G - \frac{1}{4}H^2 + \frac{1}{2}|A^\circ|^2$$

Sketch of the proof-3: nondegeneracy

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Lemma Let (M, g) be a closed Riemannian 3-manifold whose scalar curvature is strictly positive at a point:

$$\exists \bar{p} \in M : \quad R_g(\bar{p}) > 0.$$

Then for a minimizing sequence f_k of E

$$\liminf_k (\text{diam}_g\, f_k(\mathbb{S}^2)) > 0$$

where $\text{diam}_g\, f_k(\mathbb{S}^2)$ is the diameter of $f_k(\mathbb{S}^2)$ in the Riemannian manifold (M, g).

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- $\Rightarrow$ if f_k shrink then by blow up

$$\liminf E(f_k) \geq 4\pi$$

Sketch of proof-5: Existence of candidate minimizer

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- $\Rightarrow$ μ is a candidate minimizer and we have to prove regularity i.e. this measure is associated to a smooth immersion of a sphere

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- From energy bound $\Rightarrow$ only finitely many bad points

Sketch of proof-7: Graphical Decomposition Lemma

Adapting Simon setting we get Graphical decomposition in Riemannian manifolds:

IDEA: in a neighboorod of a good point, each surface is overlapping of many sheets. Each one is union of a lipschitz graph (with holes) and small "pimples"

Sketch of proof-8:Towards $C^{1,\alpha}$ regularity

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Lemma Let (M,g), f_k and μ as before. Let ξ_0 a ε_0-good point for $\varepsilon > 0$ small enough.
Then we have for all $\xi \in spt\mu \cap B_{\frac{\rho_0}{2}}(\xi_0)$ and all $\rho \leq \frac{\rho_0}{4}$ that

$$\liminf_{k\to\infty} \int_{B_{\frac{\rho}{8}}(\xi)} |A_k|^2 \, d\mu_k \leq c\rho^\alpha \quad \text{where } c = c(\rho_0) \text{ and } \alpha \in (0,1).$$

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Proof By Radon Nikodym and a modification of Poincaré inequality by Simon

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The functions u^l such that $\mu \llcorner B_\rho(\xi_0) = \sum_{l=1}^{M} \mathcal{H}^2_g \llcorner \big(\text{graph}\, u^l \cap B_\rho(\xi_0)\big)$ are $C^{1,\alpha} \cap W^{2,2}$ and satisfy the power decay

$$\int_{B_\sigma} |D^2 u^l|^2 \leq C\sigma^\alpha.$$

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- conclude by Morrey Lemma

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- Assume by contradiction there exists a bad point $\xi_0 \in spt\mu$, then by definition there exists ε_0 such that

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Sketch of the proof-11: Non existence of bad points-b

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$\Rightarrow$ equation+bootstrap gives the smoothness of the immersion

Minimization of $W_1 = \int \left(\frac{|H|^2}{4} + 1\right)$ in compact manifolds

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REMARK- the curvature conditions can be fulfilled, for instance they hold for a round sphere $\mathbb{S}^3(R)$ if $\frac{1}{\sqrt{2}} \le R < 1$,

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REMARK- the curvature conditions can be fulfilled, for instance they hold for a round sphere $\mathbb{S}^3(R)$ if $\frac{1}{\sqrt{2}} \leq R < 1$,
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Minimization of $W_1 = \int \left(\frac{|H|^2}{4} + 1 \right)$ in compact manifolds

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Minimization of Willmore type functionals in NONCOMPACT manifolds

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b) is solved by the choice of the manifold: positive curvature in some point at finite + asymptotically euclidean or hyperbolic

Asymptotic conditions

1) (M, g) is said asymptotically euclidean if there exist compact subsets $\Omega_1 \subset\subset M$ and $\Omega_2 \subset\subset \mathbb{R}^3$ such that

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$$\Rightarrow diam(\Sigma) \geq \frac{1}{C\left(\int_\Sigma |H|^p\right)^{\frac{1}{p-m}}}, \; Area(\Sigma) \geq \frac{1}{C\left(\int_\Sigma |H|^p\right)^{\frac{m}{p-m}}}$$

The articles

- A. Mondino, *Some results about the existence of critical points for the Willmore functional*, Math. Zeit., Vol. 266, Num. 3, (2010), 583-622.
- A. Mondino, *The conformal Willmore Functional: a perturbative approach*, to appear in JGA (2011).
- A. Mondino, *Existence of Integral m-Varifolds minimizing $\int |A|^p$ and $\int |H|^p$ in Riemannian Manifolds*, arXiv:1010.4514, submitted, (2010).
- E. Kuwert, A. Mondino, J. Schygulla *Existence of immersed spheres minimizing curvature functionals in compact 3-manifolds,* ArXiv: 1111.4893, submitted (2011).
- A. Mondino, J. Schygulla *Existence of immersed spheres minimizing curvature functionals in noncompact 3-manifolds,* submitted (2012)

!!THANK YOU FOR THE ATTENTION!!