Willmore spheres in Riemannian manifolds

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Introduction

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- (M, g) 3-d Riemannian manifold (later also dim$(M) \geq 3$)
- Σ closed (compact, $\partial\Sigma = \emptyset$) 2-d surface
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- A_{ij}= II fundamental form of $f(\Sigma)$
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Question

Which are the best immersions f ?

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How it is possible to relax the definitions in order to get existence?

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2-if $(M, g) = (\mathbb{R}^3, eucl)$ then by Gauss Bonnet Theorem

$$W(f) = W_{cnf}(f) + 2\pi\chi_E(\Sigma) = \frac{1}{2}E(f) + \pi\chi_E(\Sigma)$$

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W_{cnf} is conformally invariant, i.e.

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W is conformal invariant in $\mathbb{R}^3$ but not in a general manifold $\Rightarrow$ W_{cnf} is the "correct" Willmore functional from a conformal point of view.

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- Works by Bernard, Bryant, Hélein, Heller, Kilian, Mazzeo, Montiel, Pedit, Pinkall, Ritoré, Ros, Rosenberg, Schätzle, Schmidt, Schygulla, Topping, Urbano etc.

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Theorem [M.(Math. Zeit. '10)]
Assume

- $\exists \bar{p} \in \mathbb{R}^3$ such that $R_1(\bar{p}) \neq 0$,
- Said $\|h(p)\| := \sup_{|v|=1} |h_p(v,v)|$
  i) $\lim_{|p|\to\infty} \|h(p)\| = 0$.
  ii) $\exists C > 0$ and $\alpha > 2$ s.t. $|D_\lambda h_{\mu\nu}(p)| < \frac{C}{|p|^\alpha} \quad \forall \lambda, \mu, \nu = 1 \dots 3$.

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Then, for ϵ small enough, there exists a perturbed standard sphere $S_{p_\epsilon}^{\rho_\epsilon}(w_\epsilon(p_\epsilon, \rho_\epsilon))$ (where $w_\epsilon(p_\epsilon, \rho_\epsilon) \in C^{4,\alpha}(S^2)$) which is a Willmore embedding of $\mathbb{S}^2$ in $(\mathbb{R}^3, g_\epsilon)$

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- Related perturbative results, under area constraint, by Lamm-Metzger-Schulze and Lamm-Metzger

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$$\inf_{f:\mathbb{S}^2 \hookrightarrow M} W_{conf}(f) = 0.$$

- $\Rightarrow$ In Berger spheres: a minimizing sequence either converges to a totally umbilical surface (but this does not exist by S-T) or it shrinks to a point $\Rightarrow$ minimization cannot be performed $\to$ Perturbative approach, saddle type critical points.

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More precisely, every Willmore surface we construct is a normal graph over a totally umbilic sphere in $(\mathbb{S}^3, g_0)$ via a smooth function w_ε converging to 0 in $C^{4,\alpha}$ norm as $\varepsilon \to 0$.

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Remark. The critical points we construct are saddle points for W_{conf}. Moreover, a standard *bumpy-metric argument* shows that (in case $(\mathbb{S}^3, g_\varepsilon)$ *does not* have constant sectional curvature) these are *generically non-degenerate* of index exactly 4.

The Lie group case

Corollary(Carlotto-M.'13) Let $g_\varepsilon = g_0 + \varepsilon h$ be a left-invariant metric on $SU(2) \cong \mathbb{S}^3$. There exists $\overline{\varepsilon} \in \mathbb{R}_{>0}$ such that if $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$ then for every $p \in \mathbb{S}^3$ there exists an embedded critical 2-sphere for the conformal Willmore functional (in metric g_ε) passing through p. As a result, under these assumptions the conformal Willmore functional W_{conf} has uncountably many distinct critical points.

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- Question: can we find at least generalized totally geodesic immersions (i.e. minimizers of $E := \frac{1}{2}\int |A|^2$)?

Minimization of $E = \frac{1}{2}\int |A|^2$ among immersed spheres

Theorem[Kuwert- M.- Schygulla '11] Let (M, g) be a compact 3-dimensional Riemannian manifold and assume:
a) there exists $f : \mathbb{S}^2 \hookrightarrow M$ smooth immersion with $E(f) < 4\pi$

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Corollary. If (M, g) is a compact 3-dimensional Riemannian manifold with strictly positive sectional curvature, then there exists a smooth minimizer of $E = \frac{1}{2}\int |A|^2$ among smooth immersed spheres; i.e. there exists a generalized totally geodesic immersion.

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- The case (M, g) non compact is studied by M.-Schygulla '12.

Sketch of proof-1:Framework

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- Use the equation+bootstrap $\Rightarrow$ smoothness of the immersion.

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$\rightarrow$ parametric approach

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Parametric approach 2: weak, possibly branched, immersions

For any *possibly branched lipschitz immersion* we can define almost everywhere the Gauss map

$$\vec{n}_{\vec{\Phi}} := \star_h \frac{\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi} \wedge \partial_{x_2}\vec{\Phi}|} \ \in \ \wedge^{m-2} T_{\vec{\Phi}(x)} M^m$$

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Definition:[M., Rivière '11] A possibly branched lipschitz immersion $\vec{\Phi} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ is called "weak, possibly branched, immersion" if the Gauss map satisfies

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$\rightarrow$ right functional space where defining $W, W_{conf}, E, \ldots$.

Parametric approach 3: The issue of conformality

Proposition:[Toro,Müller-Sverak- Hélein,Rivière]
Let $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ then $\exists \Psi : \mathbb{S}^2 \to \mathbb{S}^2$ bilipschitz homeomorphism such that $\vec{\Phi} \circ \Psi$ is weakly conformal : almost everywhere on $\mathbb{S}^2$

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where (x_1, x_2) are local arbitrary conformal coordinates in $\mathbb{S}^2$ for the standard metric. Moreover $\vec{\Phi} \circ \Psi$ is in $W^{2,2} \cap W^{1,\infty}(\mathbb{S}^2, M^m)$.

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Remark: We don't ask conformality from the beginning for variational reasons

Parametric approach 4: Relative compactness in $\mathcal{F}_{\mathbb{S}^2}$

Theorem[M., Rivière '11] Let $\vec{\Phi}_k \in \mathcal{F}_{S^2}$ be conformal such that

$$\limsup_{k\to+\infty} \int_{\mathbb{S}^2} \left[1 + |D\vec{n}_{\vec{\Phi}_k}|_h^2\right] \, dvol_{g_k} < +\infty \quad \liminf_{k\to+\infty} \, diam(\vec{\Phi}_k(\mathbb{S}^2)) > 0. \tag{4}$$

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Moreover $\exists (f_k^i)_{i=1\cdots N} \subset \mathcal{M}^+(\mathbb{S}^2)$, for every $1 \leq i \leq N$ $\exists b^{i,1} \cdots b^{i,N^i}$ such that

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Then, up to subsequences in k, $\exists \Psi_k : \mathbb{S}^2 \to \mathbb{S}^2$ bilipschitz homeomorphism,

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Moreover $\exists (f_k^i)_{i=1\cdots N} \subset \mathcal{M}^+(\mathbb{S}^2)$, for every $1 \le i \le N$ $\exists b^{i,1} \cdots b^{i,N^i}$ such that

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Remark: related compactness, independently, by Chen-Li (2011)

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Proposition[M., Rivière '12] Let $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$, then W is Fréchet differentiable for normal $W^{1,\infty} \cap W^{2,2}$ pertubations supported away from the branched points: $spt(\vec{w}) \subset \mathbb{S}^2 \setminus \cup_{i=1}^{N} b^i$.

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Remark: Regularity for all critical points $\rightarrow$ suitable for saddle points

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$\rightarrow$ minimize $Area + W$.

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  -Under which conditions (e.g. curvature bounds in the ambient manifold) just one sphere is better?

The articles

- A. Mondino, *Some results about the existence of critical points for the Willmore functional*, Math. Zeit.,(2010).
- A. Mondino, *The conformal Willmore Functional: a perturbative approach*, Journ. Geom. Anal., (2013).
- E. Kuwert, A. Mondino, J. Schygulla *Existence of immersed spheres minimizing curvature functionals in compact 3-manifolds,* submitted, (2011).
- A. Mondino, J. Schygulla *Existence of immersed spheres minimizing curvature functionals in noncompact 3-manifolds,* submitted, (2012).
- A. Mondino, T. Rivière *Immersed Spheres of Finite Total Curvature into Manifolds,* Adv. in Calc. Var. (2013).
- A. Mondino, T. Rivière *Willmore Spheres in Compact Riemannian Manifolds,* Adv. in Math., (2013).
- A. Carlotto, A. Mondino, *Existence of generalized totally umbilic 2-spheres in perturbed 3-spheres,* submitted, (2013).

!!THANK YOU FOR THE ATTENTION!!