Min-max Theory in Geometry

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Definitions

- (M^{n+1}, g) closed $(n+1)$-Riemannian manifold with $n \leq 6$;
- $\mathcal{Z}_n(M) = \{$all oriented hypersurfaces in $M\}$;
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Question

Do minimal hypersurfaces exist?

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Theorem (Pitts, '81 $2 \leq n \leq 5$, Schoen-Simon, '81 $n = 6$)

Assume $\mathbf{L}[\Phi] > \sup_{x \in \partial X^k} vol(\Phi(x))$.

There is smooth embedded minimal n-hypersurface Σ (with multiplicities) so that

$$\mathbf{L}[\Phi] = vol(\Sigma).$$

One dimensional cycles ($k = 1$)

- $f : M \to [0, 1]$ Morse function;
- $\Phi_1 : [0, 1] \to Z_n(M), \quad \Phi_1(t) = \partial\{f < t\}$.

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Every (M^{n+1}, g) with $2 \leq n \leq 6$ admits a smooth embedded minimal hypersurface.

Application 2 (Simon-Smith '82)

Every (S^3, g) admits an embedded minimal sphere.

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Application 3 (Colding-Minicozzi, '06)

Assume (M^3, g) has $\pi_3(M) = \mathbb{Z}$ (e.g, if M is simply connected).

Ricci flow starting at g will have a finite time singularity.

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Application 4 (Marques-M., '11)

Assume (M^3, g) has $Ric(g) > 0$ and scalar curvature $R \geq 6$.

Then

$$\mathbf{L}[\Phi_1] \leq 4\pi$$

with equality only for (S^3, g_0).

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Application 5 (X. Zhou., '12)

Assume (M^{n+1}, g) has $Ric(g) > 0$ and $2 \leq n \leq 6$.

Then $\mathbf{L}[\Phi_1]$ is attained by either an index one minimal embedded hypersurface or a stable non-orientable embedded hypersurface with multiplicity two.

One, two, and three dimensional cycles ($k = 1, 2, 3$)

Consider

$$\Phi_1 : \mathbb{RP}^1 \to \mathcal{Z}_1(S^2, \mathbb{Z}_2), \quad [a_0, a_1] \mapsto \partial\{a_0 + a_1 x_1 < 0\} \cap S^2$$

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Theorem (Lusternik–Schnirelmann, ’47)

$[\Phi_1], [\Phi_2], [\Phi_3]$ are all homotopically distinct in $Z_1(S^2, \mathbb{Z}_2)$

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Every (S^2, g) admits three distinct simple closed geodesics.

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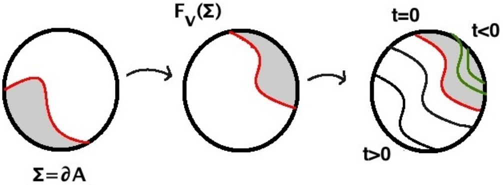
Application 2 (Jost, '89)

Every (S^3, g) admits four distinct embedded minimal spheres.

Five dimensional cycles ($k = 5$)

- B^4 = unit 4-ball $= Conf_+(S^3)/SO(4)$;
- Σ embedded smooth surface of S^3;
- $\Phi_5 : B^4 \times [-\pi, \pi] \to \mathcal{Z}_2(S^3), \quad \Phi_5(v, t) = \partial\{signed\ dist(x, F_v(\Sigma)) < t\}$.



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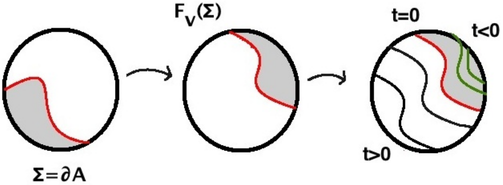

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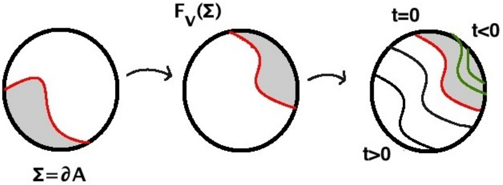

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Application (Marques-N., '12)

If $\tilde{\Sigma} \subset \mathbb{R}^3$ compact and embedded has positive genus then

$$\int_{\tilde{\Sigma}} |H|^2 d\mu \geq 2\pi^2.$$

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$$G : S^1 \times S^1 \to S^4, \quad G(s,t) = \frac{\gamma_1(t) - \gamma_2(s)}{|\gamma_1(t) - \gamma_2(t)|},$$

and set $G(\gamma_1, \gamma_2) = G(S^1 \times S^1) \in \mathcal{Z}_2(S^3)$;

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$$\Phi_5(v, \lambda) = G(F_v(\gamma_1), \lambda(F_v(\gamma_2) - c(v)) + c(v)).$$

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Application (Agol-Marques-N., '12)

If (γ_1, γ_2) has $|lk(\gamma_1, \gamma_2)| = 1$, then Mobius cross energy of $(\gamma_1, \gamma_2) \geq 2\pi^2$.

k-dimensional cycles ($k \in \mathbb{N}$)

We say $\Phi : \mathbb{RP}^k \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a k-sweepout if for every curve γ

$$0 \neq [\gamma] \in \pi_1(\mathbb{RP}^k) \implies \Phi \circ \gamma : S^1 \to \mathcal{Z}_n(M; \mathbb{Z}_2) \text{ is a sweepout.}$$

Example 1

With $f : M \to \mathbb{R}$ Morse function set

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Example 2 (Conjectural)

With $\phi_0, \ldots, \phi_k$ linearly independent eigenfunctions of Laplacian consider

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Consider for every $k \in \mathbb{N}$

$$\omega_k(M) := \inf_{\{\Phi \text{ is a } k\text{-sweepout}\}} \sup_{x \in \mathbb{RP}^k} vol(\Phi(x)).$$

Theorem (Gromov, '87 – Guth, '07)

For every (M^{n+1}, g) there is C so that for all $k \in \mathbb{N}$

$$C^{-1} k^{\frac{1}{n+1}} \leq \omega_k(M) \leq C k^{\frac{1}{n+1}}.$$

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- (Franks, '92, Bangert '93) (S^2, g) admits infinitely closed geodesics
- (Yau's Conjecture, '82) (M^3, g) admits infinitely many distinct smooth minimal surfaces.

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4: finitely many minimal hypersurfaces + **2** + **3** $\implies \omega_k(M)$ grows linearly.

Contradiction with Gromov-Guth Theorem!