Surfaces making constant angle with certain vector fields in 3-spaces

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Outline

Constant Angle Surfaces

A constant angle surface (**CAS** in short) is an oriented surface for which its normal makes a constant angle with a fixed direction, which is chosen in each case as a preferred direction in the ambient space:

1. $\mathbb{R}$ direction in $\mathbb{M}^2 \times \mathbb{R}$, $\mathbb{M}^2 \times \mathbb{R}_1$
2. position vector in $\mathbb{E}^3$ and $\mathbb{E}^3_1$
3. Killing vector field in $\mathbb{E}^3$
4. Killing vector field $e_3 = \partial_z$ in Nil_3
5. left invariant vector field in $G(\mu_1, \mu_2)$, in particular in $\mathrm{Sol}_3 = G(-1, 1)$

Constant Angle Surfaces and Principal Directions

A property of CAS:
When the ambient is of the form $\mathbb{M}^2 \times \mathbb{R}$, a favored direction is $\mathbb{R}$. It is known that for a constant angle surface in $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or in $\mathbb{H}^2 \times \mathbb{R}$, the projection of $\frac{\partial}{\partial t}$ (where t is the global parameter on $\mathbb{R}$) onto the tangent plane of the immersed surface, denoted by T, is a principal direction[1] with the corresponding principal curvature[2] identically zero.

Study surfaces endowed with a principal direction T which will be called a **canonical principal direction** (**CPD** in short).

[1] eigenvector of the shape operator
[2] eigenvalue of the shape operator

Results - CAS in $\mathbb{M}^2 \times \mathbb{R}$

F. Dillen, J. Fastenakels, J. Van der Veken, L. Vrancken, *Constant angle suraces in* $\mathbb{S}^2 \times \mathbb{R}$, Monatsh. Math. **152** (2)(2007), 89–96.

F. Dillen, M.I. Munteanu, *Constant Angle Surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Bull. Braz. Math. Soc. **40** (1) (2009) 1, 85–97.

M.I. Munteanu, N., *A new approach on constant angle surfaces in* $\mathbb{E}^3$, Turkish J. Math. **33** (2) (2009), 169–178.

Results - CPD in $\mathbb{M}^2 \times \mathbb{R}$

- F. Dillen, J. Fastenakels, J. Van der Veken, *Surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with a canonical principal direction*, Ann. Glob. Anal. Geom., 35(2009) 4, 381–396.
- F. Dillen, M.I. Munteanu, N., *Surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with a canonical principal direction*, Taiwanese J. Math., 15 (2011) 5, 2265-2289.
- M.I. Munteanu, N., *Complete classification of surfaces with a canonical principal direction in the Euclidean space $\mathbb{E}^3$*, Cent. Eur. J. Math., 9(2011)2, 378–389.

CAS&CPD in $\mathbb{M}^2 \times \mathbb{R}_1$ - Preliminaries

- $\mathbb{M}^2 \times \mathbb{R}_1$ the product of $\mathbb{M}^2(c)$ - a 2-dimensional space form of constant sectional curvature c and $\mathbb{R}_1$
- $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{\mathbb{M}^2} - dt^2$ - the Lorentzian metric
- the surface M is given by $L : M \to \mathbb{M}^2 \times \mathbb{R}_1$
- ξ is a δ-unit normal vector field to M, $\langle \xi, \xi \rangle = \delta \in \{-1, 1\}$.
  - $\delta = -1$, ξ is timelike iff M is spacelike
  - $\delta = 1$, ξ is spacelike iff M is timelike
- $\partial_t := (\partial/\partial t),\ \ t \in \mathbb{R}_1$ is a unitary timelike vector field on $\mathbb{M}^2 \times \mathbb{R}_1$,

$$\partial_t = T + \Theta\xi$$

$\Theta = \cosh\theta$ (M is spacelike) or $\Theta = \sinh\theta$ (M is timelike).

- for every tangent vector field X on M:

$$\nabla_X T = \Theta S X, \tag{1}$$

$$X(\Theta) = -\delta\langle SX, T\rangle. \tag{2}$$

Preliminaries - more about angles

Recall that ∂_t is timelike in $(\mathbb{M}^2 \times \mathbb{R}_1, \langle\ ,\ \rangle)$.
If M is a spacelike surface, then ξ is timelike, and we will consider it with the same time-orientation as ∂_t, i.e. future-directed, $\langle \partial_t, \xi \rangle < 0$.

- The angle θ between two unitary timelike vectors was defined for the first time in 1984, as:

$$\cosh\theta = -\langle \partial_t, \xi \rangle.$$

If M is a timelike surface, then ξ is spacelike.

- The angle θ between a timelike and a spacelike unitary vectors was defined for the first time in 2005, as:

$$\sinh\theta = \langle \partial_t, \xi \rangle.$$

G. Birman, K. Nomizu, *Trigonometry in Lorentzian geometry*, Amer. Math. Monthly, 91(1984)9, 543–549.

E. Nešović, M.Petrović - Torgašev, L. Verstraelen, *Curves in Lorentzian spaces*, Boll. Unione Mat. Ital., serie 8, 8-B(2005)3, 685–696.

Spacelike surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ and $\mathbb{H}^2 \times \mathbb{R}_1$

• constant angle spacelike surfaces in Minkowski 3−space (when $c = 0$) have been classified in

R. López, M.I. Munteanu, *Constant angle surfaces in Minkowski space*, Bull. Belg. Math. Soc. - Simon Stevin , 18 (2011) 2, 271–286.

• we extend this study to constant angle spacelike surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ ($c = 1$) and $\mathbb{H}^2 \times \mathbb{R}_1$ ($c = -1$) .

• we study canonical principal directions for spacelike surfaces in the ambient space $\mathbb{M}^2 \times \mathbb{R}_1$.

Spacelike surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ and $\mathbb{H}^2 \times \mathbb{R}_1$

Theorem (Fu, N.)

Let M be a spacelike surface immersed into Lorentzian product space $\mathbb{M}^2(c) \times \mathbb{R}_1$, with $c = -1, 1$. Then, M is a constant angle spacelike surface if and only if the immersion L is locally given by:

(a) If $c = 1$, then $L : M \to \mathbb{S}^2 \times \mathbb{R}_1 \hookrightarrow \mathbb{R}^4_1$,
$L(u, v) = \big(\cos(u \cosh\theta) f(v) + \sin(u \cosh\theta) f(v) \times f'(v), -u \sinh\theta\big)$,
where f is a unit speed curve in $\mathbb{S}^2$.

(b) If $c = -1$, then $L : M \to \mathbb{H}^2 \times \mathbb{R}_1 \hookrightarrow \mathbb{R}^4_2$,
$L(u, v) = \big(\cosh(u \cosh\theta) f(v) + \sinh(u \cosh\theta) f(v) \boxtimes f'(v), -u \sinh\theta\big)$,
where f is a unit speed curve in $\mathbb{H}^2$.

In both cases $\theta \neq 0$ denotes the constant hyperbolic angle.

If $\theta = 0$, then ∂_t is a normal vector field and M is an open part of the spacelike surface:
- $\mathbb{S}^2 \times \{t_0\}$ for $c = 1$
- $\mathbb{H}^2 \times \{t_0\}$ for $c = -1$.

Spacelike surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ and $\mathbb{H}^2 \times \mathbb{R}_1$

Theorem (Fu, N.)

Let $L : M \to \mathbb{M}^2(c) \times \mathbb{R}_1,\ c = -1, 1$ be a spacelike surface and let θ be the hyperbolic angle function. Then, T is a canonical principal direction for M if and only if M is parameterized as:

(a) If $c = 1$, then $L : M \to \mathbb{S}^2 \times \mathbb{R}_1 \hookrightarrow \mathbb{R}^4_1$,
$L(u,v) = \Big(\cos\phi(u)\ f(v) + \sin\phi(u)\ N_f(v), \chi(u)\Big)$, where f is a regular curve on $\mathbb{S}^2$ and $N_f(v) = \frac{f(v)\times f'(v)}{\sqrt{\langle f'(v), f'(v)\rangle}}$ is the normal of f.

(b) If $c = -1$, then $L : M \to \mathbb{H}^2 \times \mathbb{R}_1 \hookrightarrow \mathbb{R}^4_2$,
$L(u,v) = \Big(\cosh\phi(u)\ f(v) + \sinh\phi(u)\ N_f(v), \chi(u)\Big)$, where f is a regular curve on $\mathbb{H}^2$ and $N_f(v) = \frac{f(v)\boxtimes f'(v)}{\sqrt{\langle f'(v), f'(v)\rangle}}$ is the normal of f.

Moreover, $\phi(u) = \int^u \cosh\theta(\tau)d\tau$ and $\chi(u) = -\int^u \sinh\theta(\tau)d\tau$.

CPD for spacelike surfaces in $\mathbb{E}^3_1$

Theorem (N.)

Let $L : M \to \mathbb{E}^3_1$ be a spacelike surface isometrically immersed in $\mathbb{E}^3_1$ and let $\theta(p) \neq 0$ be the hyperbolic angle function. Then, M has a canonical principal direction if and only if M is parametrized by:

(c.1) $L(u,v) =$

$$\big(\cos v, \sin v, 0\big) \int^{u} \cosh\theta(\tau)d\tau - (0,0,1) \int^{u} \sinh\theta(\tau)d\tau + \gamma(v),$$

where

$$\gamma(v) = \Big(\int \psi(v)\sin v \; dv, -\int \psi(v)\cos v \; dv, \; 0\Big), \psi \in C^\infty(M),$$

(c.2) $L(u,v) =$

$$\big(\cos v_0, \sin v_0, 0\big) \int^{u} \cosh\theta(\tau)d\tau - (0,0,1) \int^{u} \sinh\theta(\tau)d\tau + v\gamma_0,$$

where $\gamma_0 = \Big(-\sin v_0, \; \cos v_0, \; 0\Big)$, and v_0 is a real constant.

CPD for spacelike surfaces in $\mathbb{E}^3_1$ with $H = 0$

Theorem (N.)

The only maximal spacelike surfaces in $\mathbb{E}^3_1$ with a canonical principal direction are the catenoids of 1st kind, $L : M \longrightarrow \mathbb{E}^3_1$,

$$L(u,v) = \left(\sqrt{u^2 - m^2}\cos v,\ \sqrt{u^2 - m^2}\sin v,\ m\ln(u + \sqrt{u^2 - m^2})\right),$$

$m \in \mathbb{R}^*$.

Remark

Under the assumption of flatness, we obtain the generalized cylinders from case (c.2) of the classification theorem.

The catenoid of 1st kind

The catenoid of 1st kind may be obtained rotating the curve

$$\left(m \sinh\left(\frac{t}{m} - \ln m\right),\ 0,\ t\right)$$

around the Oz axis.

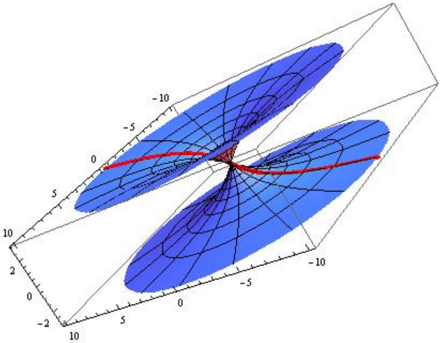


Figure: $m = 1,\ t \in [-3, 3],\ v \in [0, 2\pi]$

Timelike surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ and $\mathbb{H}^2 \times \mathbb{R}_1$

• constant angle timelike surfaces in Minkowski 3−space (when $c = 0$) have been classified in

F. Güler, G. Şaffak, E. Kasap, *Timelike constant angle surfaces in Minkowski space* $\mathbb{R}^3_1$, Int. J. Contemp. Math. Sci., 6(2011)44, 2189–2200.

• we study constant angle timelike surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ ($c = 1$) and $\mathbb{H}^2 \times \mathbb{R}_1$ ($c = -1$) .

• we study canonical principal directions for timelike surfaces in the ambient space $\mathbb{M}^2 \times \mathbb{R}_1$.

Timelike CAS $\mathbb{S}^2 \times \mathbb{R}_1$ and $\mathbb{H}^2 \times \mathbb{R}_1$

Theorem (Fu, N.)

Let $L : M \to \mathbb{M}^2(c) \times \mathbb{R}_1$ be a timelike surface immersed into Lorentzian product space $\mathbb{M}^2(c) \times \mathbb{R}_1$. Then M is a constant angle timelike surface if and only if the immersion L is locally given by:

(a) If $c = 1$, then $L : M \to \mathbb{S}^2 \times \mathbb{R}_1$,
$L(u, v) = \big(\cos(u \sinh\theta) f(v) + \sin(u \sinh\theta) f(v) \times f'(v),\ u \cosh\theta \big)$,
where f is a unit speed curve in $\mathbb{S}^2$,

(b) If $c = -1$, then $L : M \to \mathbb{H}^2 \times \mathbb{R}_1$,
$L(u, v) = \big(\cosh(u \sinh\theta) f(v) + \sinh(u \sinh\theta) f(v) \boxtimes f'(v),\ u \cosh\theta \big)$,
where f is a unit speed curve in $\mathbb{H}^2$.

In both cases $\theta \neq 0$ denotes the constant hyperbolic angle.

If $\theta = 0$, then ∂_t is a tangent vector field, has no normal component, and M is an open part of $\gamma \times \mathbb{R}_1$, where $\gamma \in \mathbb{M}^2(c)$, $c \in \{-1, 1\}$.

CPD for timelike surfaces in $\mathbb{M}^2 \times \mathbb{R}_1$

Theorem (Fu, N.)

Let $L : M \to \mathbb{M}^2(c) \times \mathbb{R}_1$ be a timelike surface and let θ be the hyperbolic angle function. Then, T is a canonical principal direction for M if and only if M is parametrized as:

(a) If $c = 1$, then $L : M \to \mathbb{S}^2 \times \mathbb{R}_1$,
$L(u,v) = \Big(\cos\chi(u)\ f(v) + \sin\chi(u)\ N_f(v),\ \phi(u)\Big)$, where f is a regular curve on $\mathbb{S}^2$ and $N_f(v) = \frac{f(v)\times f'(v)}{\sqrt{\langle f'(v), f'(v)\rangle}}$ is the normal of f.

(b) If $c = -1$, then $L : M \to \mathbb{H}^2 \times \mathbb{R}_1$,
$L(u,v) = \Big(\cosh\chi(u)\ f(v) + \sinh\chi(u)\ N_f(v),\ \phi(u)\Big)$, where f is a regular curve on $\mathbb{H}^2$ and $N_f(v) = \frac{f(v)\boxtimes f'(v)}{\sqrt{\langle f'(v), f'(v)\rangle}}$ is the normal of f.

Moreover, $\phi(u) = \int^u \cosh\theta(\tau)d\tau$ and $\chi(u) = \int^u \sinh\theta(\tau)d\tau$.

CPD for timelike surfaces in $\mathbb{M}^2 \times \mathbb{R}_1$

Theorem (Fu, N.)

Let $L : M \to \mathbb{M}^2(c) \times \mathbb{R}_1$ be a timelike surface and let θ be the hyperbolic angle function. Then, T is a canonical principal direction for M if and only if M is parametrized as:

(c) If $c = 0$, then $L : M \to \mathbb{E}^3_1$,

(c.1) $L(u, v) = \Big(\chi(u)\cos v,\ \chi(u)\sin v,\ \phi(u)\Big) + \gamma(v)$,

where $\gamma(v) = \Big(-\int \psi(v)\sin v\ dv,\ \int \psi(v)\cos v\ dv,\ 0\Big)$,

and ψ is a smooth function,

(c.2) $L(u, v) = \Big(\chi(u)\cos v_0,\ \chi(u)\sin v_0,\ \phi(u)\Big) + \gamma_0 v$,

where $\gamma_0 = \big(-\sin v_0,\ \cos v_0,\ 0\big)$, and v_0 is a real constant.

Moreover, $\phi(u) = \int^u \cosh\theta(\tau)d\tau$ and $\chi(u) = \int^u \sinh\theta(\tau)d\tau$.

CPD for timelike surfaces in $\mathbb{E}^3_1$ - minimality

Corollary (Fu, N.)

The only flat timelike surfaces M immersed in $\mathbb{E}^3_1$ endowed with a canonical principal direction are given by the cylindrical surfaces parametrized in case **(c.2)** of previous Theorem.

Theorem (Fu, N.)

The only minimal timelike surfaces M immersed in $\mathbb{E}^3_1$ endowed with a canonical principal direction are given by the catenoids of 3rd kind parametrized as:

$$L(t, v) = \left(m \cos \frac{t}{m} \cos v, m \cos \frac{t}{m} \sin v, t \right), \quad m \in \mathbb{R}^*.$$

The catenoid of 3rd kind...

... may be obtained rotating the curve:

$$\left(m \cos \frac{t}{m}, 0, t\right), \quad m \in \mathbb{R}^*,$$

around the Oz axis.

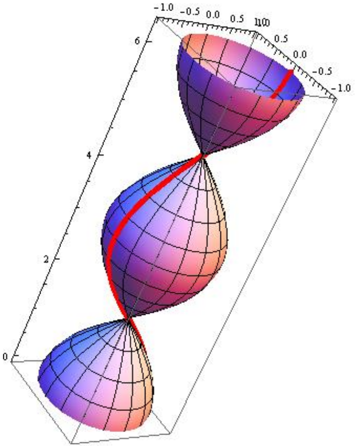


Figure: $m = 1,\ t \in [0, 2\pi],\ v \in [0, 2\pi]$

Preprints

The problems of
- CAS for spacelike and timelike surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ and $\mathbb{H}^2 \times \mathbb{R}_1$
- CPD for spacelike and timelike surfaces in $\mathbb{M}^2 \times \mathbb{R}_1$

are studied in :

Y. Fu, N. *Constant angle property and canonical principal directions for surfaces in* $\mathbb{M}^2(c) \times \mathbb{R}_1$, preprint 2012.

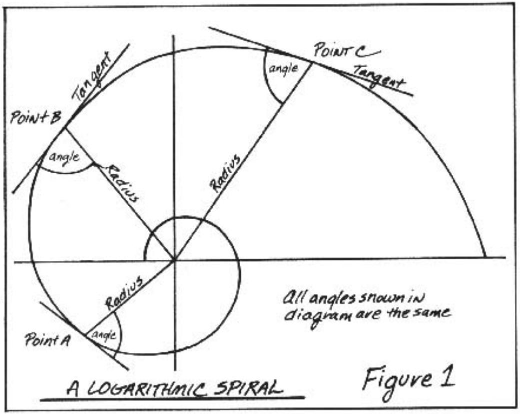
- CPD for spacelike surfaces in $\mathbb{E}^3_1$:

N., *A note on spacelike surfaces in Minkowski 3-space*, preprint 2011.

Constant angle with the position vector

Logarithmic spirals $\Longrightarrow$ constant slope surfaces

Logarithmic spiral: planar curve having the property that the angle θ between its tangent and the radial direction at every point is constant.



Constant angle with the position vector

In other words, the logarithmic spiral is the curve whose tangent makes a constant angle θ with the position vector in every point.

Question - surfaces:

Passing from curves to surfaces, find all surfaces in the Euclidean 3-space making a constant angle with the position vector.

Constant angle with the position vector

Answer: constant slope surfaces

Theorem (Munteanu)

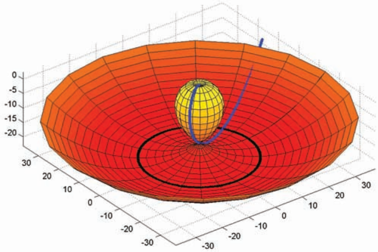
Let $r : M \longrightarrow \mathbb{E}^3$ be an isometric immersion. Then M is of constant slope if and only if either it is **an open part of the Euclidean** 2-**sphere centered in the origin**, or it can be parameterized by

$$r(u,v) = u \sin\theta \big(\cos\varphi(u)\ f(v) + \sin\varphi(u)\ f(v) \times f'(v) \big)$$

where θ is a constant (angle) different from 0, $\varphi(u) = \cot\theta \log u$ and f is a unit speed curve on the Euclidean sphere $\mathbb{S}^2$.

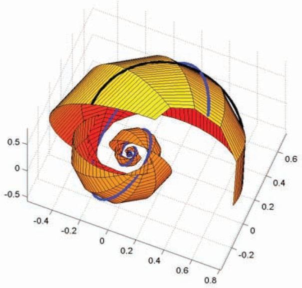
- $\theta = 0$: the position vector is normal to the surface, M is an open part of the Euclidean 2-sphere;
- $\theta = \frac{\pi}{2}$: M is a cone with the vertex in origin, or a plane passing through origin.

Examples



$\theta = \frac{\pi}{5}$

$f(v) = (\cos v, \sin v, 0)$



$\theta = \frac{\pi}{15}$

$f(v) = (\cos^2 v, \cos v \sin v, \sin v)$

parametric lines: blue: logarithmic spiral
black: the spherical curve f.

Constant slope surfaces

- M.I. Munteanu, *From Golden Spirals to Constant Slope Surfaces*, J. Math.Phys., 51 (2010) 7, 073507:1–9.
- Y. Fu, D. Yang, *On constant slope spacelike surfaces in 3-dimensional Minkowski space*, J. Math. Analysis Appl., 385 (2012) 1, 208–220.
- Y. Fu, X. Wang, *Classification of Timelike Constant Slope Surfaces in 3-Dimensional Minkowski Space*, Res. Math. 2012.

Constant angle with a Killing vector field - Preliminaries

- $\mathbb{E}^3 = (\mathbb{R}^3, \langle\ ,\ \rangle)$,
- $\overset{\circ}{\nabla}$ - Levi-Civita connection corresponding to $\langle\ ,\ \rangle$ in $\mathbb{E}^3$,
- V is Killing iff it satisfies the Killing equation:

$$\langle \overset{\circ}{\nabla}_X V, Y\rangle + \langle \overset{\circ}{\nabla}_Y V, X\rangle = 0,$$

for any vector fields X, Y in $\mathbb{R}^3$.

- The set

$$\{\partial_x,\ \partial_y,\ \partial_z,\ -y\partial_x + x\partial_y,\ z\partial_y - y\partial_z,\ z\partial_x - x\partial_z\}$$

gives a basis of Killing vector fields in $\mathbb{E}^3$.

Curves - constant angle with a Killing field

• If $\tilde{\gamma}$ is a straight line, then γ is a *helix*.
W.l.o.g. the line can be taken to be (parallel with) one of the coordinate axes, and this is an integral curve of a Killing vector vector field in $\mathbb{E}^3$.

Motivated by this remark, a natural question appears:
• *which curves make a constant angle with a Killing vector field in* $\mathbb{E}^3$?

Curves - constant angle with a Killing field

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Motivated by this remark, a natural question appears:
• *which curves make a constant angle with a Killing vector field in $\mathbb{E}^3$*?

Recall that we have a basis of Killing vector fields in $\mathbb{E}^3$:

$$\{\partial_x,\ \partial_y,\ \partial_z,\ -y\partial_x + x\partial_y,\ z\partial_y - y\partial_z,\ z\partial_x - x\partial_z\}$$

Curves - constant angle with a Killing field

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Curves - constant angle with a Killing field

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Motivated by this remark, a natural question appears:
• *which curves make a constant angle with a Killing vector field in $\mathbb{E}^3$?*

Recall that we have a basis of Killing vector fields in $\mathbb{E}^3$:

$$\{\partial_x,\ \partial_y,\ \partial_z,\ -y\partial_x + x\partial_y,\ z\partial_y - y\partial_z,\ z\partial_x - x\partial_z\}$$

• *which curves make a constant angle with V in $\mathbb{E}^3$?*

Plane curves - constant angle with V

Theorem (Munteanu, N.)

A curve in the xy-plane makes constant angle θ with the Killing vector field $V = -y\partial_x + x\partial_y$ if and only of it is given by one of the following cases:

(a) or a straight line passing through the origin,

(b) either the circle $\mathbb{S}^1(r_0)$ centred in the origin and of radius r_0,

(c) or the logarithmic spiral $\rho(\phi) = e^{\tan\theta\ (\phi-\phi_0)}$.

Sketch of proof.

For the curve p.a.l. $\gamma(s) = (\rho(s)\cos\phi(s),\ \rho(s)\sin\phi(s))$, the constant angle condition becomes: $\rho(s)\phi'(s) = \cos\theta$.

- if $\theta = \frac{\pi}{2}$: $\rho(s) \neq 0$ and $\phi(s) = \phi_0$, **(a)**,
- if $\theta = 0$: $\rho(s) = \rho_0$ and $\phi(s) = \frac{s}{\rho_0} + \phi_0$, **(b)**,
- if $\theta \neq 0$: $\rho(s) = s\sin\theta + s_0$ and $\phi(s) = \cot\theta\ln(s\sin\theta + s_0) + \phi_0$, **(c)**.

Space curves - constant angle with V

In cylindrical coordinates: $\gamma(s) = \Big(\rho(s)\cos\phi(s),\ \rho(s)\sin\phi(s),\ z(s)\Big)$

Theorem (Munteanu, N.)

A curve γ in the Euclidean space $\mathbb{E}^3 \setminus Oz$ makes a constant angle θ with the Killing vector field $V = -y\partial_x + x\partial_y$ if and only if, is given, in cylindrical coordinates (ρ, ϕ, z), up to vertical translations and rotations around z-axis, by: $\rho(s) = \rho_0 + \sin\theta \int^s \cos\omega(\zeta)d\zeta,\ \phi(s) = \cos\theta \int^s \frac{d\zeta}{\rho(\zeta)}$,

$$z(s) = \sin\theta \int^s \sin\omega(\zeta)d\zeta,$$

where $\rho_0 \in \mathbb{R}$ and ω is a smooth function on $I \subset \mathbb{R}$.

Examples for different values of ω

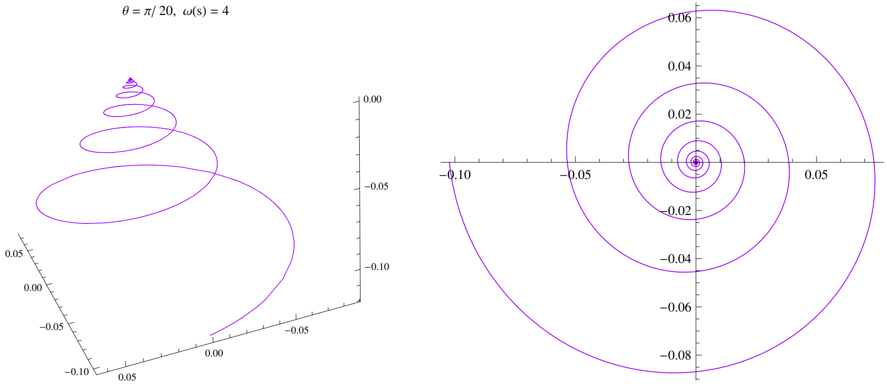


Figure: Space curve making constant angle with V (left) and its projection (right): $\omega = \omega_0$

Examples for different values of ω

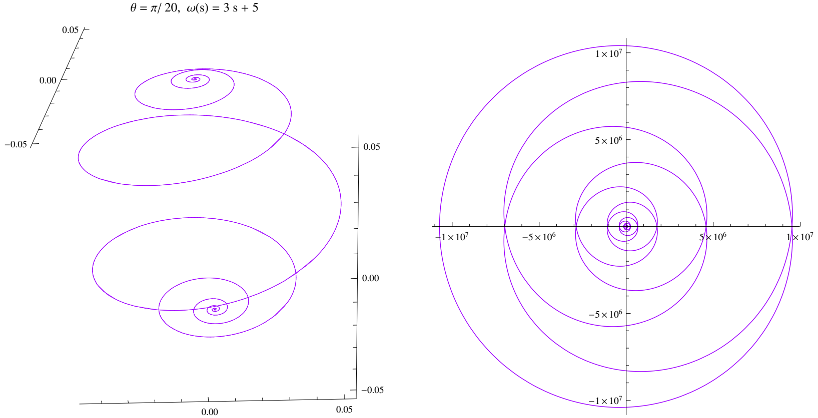
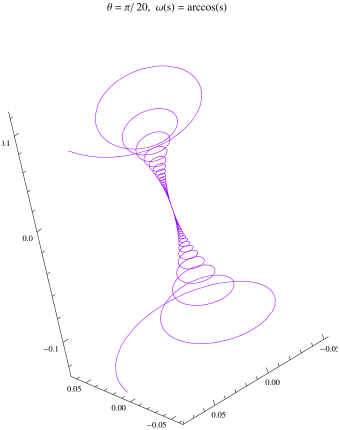


Figure: Space curve making constant angle with V (left) and its projection (right): $\omega = ms + n$

Examples for different values of ω



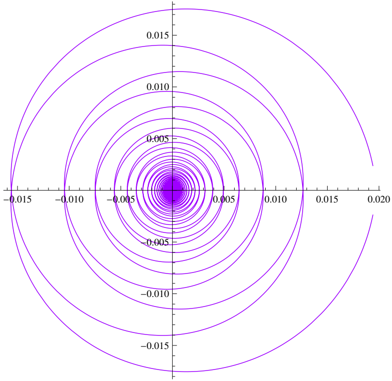


Figure: Space curve making constant angle with V (left) and its projection (right): $\omega = \arccos(s)$

Surfaces - Preliminaries

- $V = -y\partial_x + x\partial_y$ must be non-null, thus the surface M lies in $\mathbb{E}^3 \setminus Oz$;
- g the metric on M and by ∇ the associated Levi-Civita connection,
- N denotes the unit normal to the surface M;
- denote $\angle(V, N) := \theta$ - constant angle;
  - If $\theta = \frac{\pi}{2}$, then M is a surface of revolution.
  - If $\theta = 0$, then we obtain half-planes having $z-$axis as boundary.
- projecting V on the tangent plane to M:

$$V = T + \mu \cos\theta \xi,$$

where ξ is the unit normal to M, T is the tangent part, with $\|T\| = \mu \sin\theta$ and $\mu = \|V\|$.

- choose an orthonormal basis $\{e_1, e_2\}$ on the tangent plane to M s.t. $e_1 = \frac{T}{\|T\|}$ and $e_2 \perp e_1$.
- If follows that $V = \mu(\sin\theta e_1 + \cos\theta\xi)$.

Surfaces - Basic formulas

For an arbitrary vector field X in $\mathbb{E}^3$, we have

$$\overset{\circ}{\nabla}_X V = k \times X, \tag{3}$$

where $k = (0,0,1)$ and $\times$ stands for the usual cross product in $\mathbb{E}^3$.
Consider $\{e_1, e_2, k, \xi\}$ in a point on M and define the angles:

$$\angle(\xi, k) := \varphi, \quad \angle(e_1, k) := \eta \quad \angle(e_2, k) := \psi,$$

which are not independent, $\cos\varphi = -\sin\theta\sin\psi \;\; \cos\eta = \cos\theta\sin\psi$.
We decompose $k \times e_1$ and $k \times e_2$ in the basis $\{e_1, e_2, \xi\}$,

$$k \times e_1 = -\sin\theta\sin\psi\ e_2 - \cos\psi\xi,\ \ k \times e_2 = \sin\theta\sin\psi\ e_1 + \cos\theta\sin\psi\xi. \tag{4}$$

If X is tangent to M, then

$$\begin{aligned}\overset{\circ}{\nabla}_X V &= X(\mu)\Big(\sin\theta e_1 + \cos\theta\xi\Big) \\ &\quad + \mu\sin\theta\Big(\nabla_X e_1 + h(X, e_1)\Big) - \mu\cos\theta AX.\end{aligned} \tag{5}$$

Surfaces - Basic formulas

From (3), (5) and (4) we get

$$e_1(\mu) = -\cos\theta\cos\psi, \quad e_2(\mu) = \sin\psi.$$

As a consequence, we obtain the shape operator:

$$S = \begin{pmatrix} -\frac{\sin\theta\cos\psi}{\mu} & 0 \\ 0 & \lambda \end{pmatrix}, \tag{6}$$

where λ is a smooth function on M, and the Levi-Civita connection:

$$\nabla_{e_1} e_1 = -\frac{\sin\psi}{\mu}\ e_2, \quad \nabla_{e_1} e_2 = \frac{\sin\psi}{\mu}\ e_1,$$
$$\nabla_{e_2} e_1 = \lambda\ \mathrm{cotan}\theta\ e_2, \quad \nabla_{e_2} e_2 = -\lambda\ \mathrm{cotan}\theta\ e_1.$$

From (6) we see that e_1 and e_2 are principal directions on M.

Surfaces - Basic formulas

Then, using the expressions of the Levi-Civita connection we may compute the Lie bracket of e_1 and e_2:

$$[e_1, e_2] = \frac{\sin\psi}{\mu} e_1 - \lambda \ \mathrm{cotan}\theta \ e_2. \tag{7}$$

Consequently, a compatibility condition is found computing $[e_1, e_2](\mu)$ in two ways:

$$-\cos\psi \ e_1(\psi) + \cos\theta \sin\psi \ e_2(\psi) = \frac{\cos\theta \sin\psi \cos\psi}{\mu} + \lambda \ \mathrm{cotan}\theta \sin\psi. \tag{8}$$

Coordinates $(x, y, z) \mapsto (\rho\cos\phi, \rho\sin\phi, z)$

From now on we use cylindrical coordinates, such that the parametrization of the surface M may be thought as

$$F : D \subset \mathbb{R}^2 \longrightarrow \mathbb{E}^3 \setminus Oz, \quad (u, v) \mapsto \Big(\rho(u, v),\ \phi(u, v),\ z(u, v)\Big). \tag{9}$$

The Euclidean metric in $\mathbb{E}^3$ becomes a warped metric
$\langle\ ,\ \rangle = d\rho^2 + dz^2 + \rho^2 d\phi^2$

Note that the Killing vector field V coincides with ∂_ϕ.

The basis $\{e_1, e_2, \xi\}$ may be expressed in terms of the new coordinates as:

$$\begin{aligned} e_1 &= -\cos\theta\cos\psi\ \partial_\rho + \frac{\sin\theta}{\mu}\ \partial_\phi + \cos\theta\sin\psi\ \partial_z, \\ e_2 &= \sin\psi\ \partial_\rho + \cos\psi\ \partial_z, \\ \xi &= \sin\theta\cos\psi\ \partial_\rho + \frac{\cos\theta}{\mu}\ \partial_\phi - \sin\theta\sin\psi\ \partial_z. \end{aligned} \tag{10}$$

Classification theorem

Theorem (Munteanu, N.)

Let M be a surface isometrically immersed in $\mathbb{E}^3 \setminus Oz$ and consider the Killing vector field $V = -y\partial_x + x\partial_y$. Then M makes a constant angle θ with V if and only if is one of the following surfaces, up to vertical translations and rotations about $z-$axis:

(a) a half-plane with $z-$axis as boundary,

(b) a rotational surface around $z-$axis,

Classification theorem

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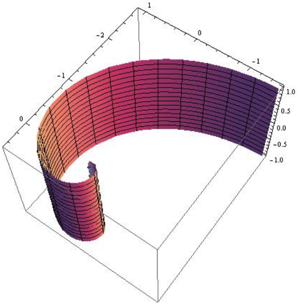
(c) a right cylinder over a logarithmic spiral given by :

$$F(u,z) = \left(u\cos\theta,\ \log(cu^{-\tan\theta}),\ z\right),$$
$c \in \mathbb{R}^*$

For $\theta = \frac{\pi}{3}$; and $c = 3$
we get this figure:



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(d) the Dini's surface defined in cylindrical coordinates (ρ, ϕ, z) by

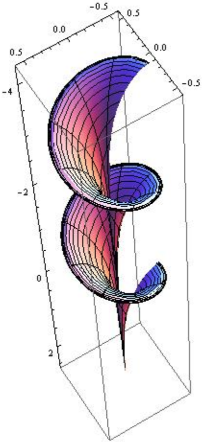
$$F(u,v) = \left(-\frac{\cos\theta\sin(cu)}{c},\ -\frac{cv\tan\theta}{\cos\theta} - \tan\theta\log\left(\tan\frac{cu}{2}\right),\right.$$
$$\left. v - \frac{\cos\theta\cos(cu)}{c}\right), \quad (11)$$

where c is a nonzero real constant.

Dini's surface - parametrization...

... from cylindrical back to Euclidean coordinates

$$F(u,v) = \left(-\frac{\cos\theta\sin(cu)}{c}\cos\left(-\frac{cv\tan\theta}{\cos\theta} - \tan\theta\log\left(\tan\frac{cu}{2}\right)\right),\right.$$
$$-\frac{\cos\theta\sin(cu)}{c}\sin\left(-\frac{cv\tan\theta}{\cos\theta} - \tan\theta\log\left(\tan\frac{cu}{2}\right)\right),$$
$$\left. v - \frac{\cos\theta\cos(cu)}{c}\right)$$



$cu \mapsto u_1$ and $-\frac{cv\tan\theta}{\cos\theta} - \tan\theta\log\left(\tan\frac{cu}{2}\right) \mapsto u_2$

$$x(u_1,u_2) = -\frac{\cos\theta}{c}\sin u_1\cos u_2,$$
$$y(u_1,u_2) = -\frac{\cos\theta}{c}\sin u_1\sin u_2,$$
$$z(u_1,u_2) = -\frac{\cos\theta}{c}\left(\cos u_1 + \log\left(\tan\frac{u_1}{2}\right)\right) - \frac{\cos\theta}{c\tan\theta}u_2.$$

Figure: $\theta = \frac{\pi}{3}$, $c = \frac{\sqrt{3}}{2}$

More over Dini's surface

- This surface is named after Ulisse Dini (1845 - 1918), who obtained it studying helicoidal surfaces.
- Dini's surface is a helicoidal surface with axis Oz:

$$F(\rho,\phi) = \big(\rho\cos\phi,\ \rho\sin\phi,\ \mathtt{h}\phi + \Lambda(\rho)\big),$$

where $(\Lambda\circ\rho)(u) = -\frac{\cos\theta}{c}\left(\log\left(\tan\frac{cu}{2}\right) + \cos(cu)\right)$
and the *pitch* equals to $\mathtt{h} = -\frac{\cos\theta}{c\tan\theta}$.

- It may be obtained *twisting the pseudosphere* of radius $\frac{\cos\theta}{c}$.
- It has constant negative Gaussian curvature depending on the constant angle θ, $K = -c^2\tan^2\theta,\ c\in\mathbb{R}^*$.

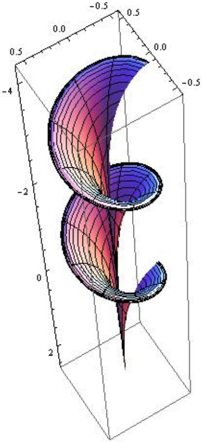


Figure: $\theta = \frac{\pi}{3}$, $c = \frac{\sqrt{3}}{2}$

Final remarks

Proposition (Munteanu, N.)

The parametric curves of Dini's surface are circular helices(v-param) and spherical curves($u-$param).

Corollary (Munteanu, N.)

Looking backward, the $u-$parameter curves make the constant angle $\frac{\pi}{2}-\theta$ with the Killing vector field V and the affine function ω (appearing in Theorem of space curves) is given by $\omega(s)=cs,\ c\in\mathbb{R}^*$.

Let M make a constant angle with the Killing vector field V, and:

- M is totally geodesic iff it is a vertical plane with the boundary Oz;
- M is minimal not totally geodesic iff it is a catenoid about $z-$axis;
- M is flat iff it is a vertical plane with the boundary $z-$axis, a flat rotational surface or a right cylinder over a logarithmic spiral.

Final remarks

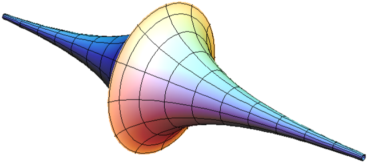
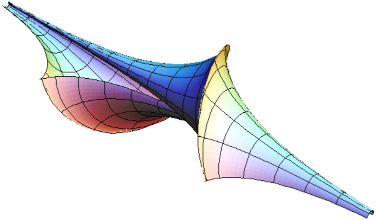


Figure: Pseudosphere



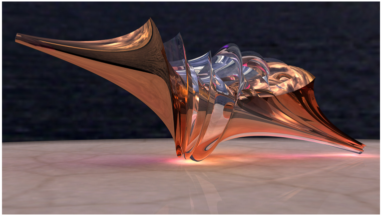


Figure: Dini's surface

Source: http://virtualmathmuseum.org/galleryS.html

Articles

The problems of
- curves making constant angle with a rotational Killing vector field in $\mathbb{E}^3$
- surfaces making constant angle with a rotational Killing vector field in $\mathbb{E}^3$
are studied in :

M.I. Munteanu, N. *Surfaces in $\mathbb{E}^3$ making constant angle with Killing vector fields*, Internat. J. Math., 23 (2012) 6, 1250023:1-16.

CAS in Heisenberg group Nil_3

$$\mathbb{R}^3: \quad (\mathbf{x},\mathbf{y},\mathbf{z}) * (\bar{\mathbf{x}},\bar{\mathbf{y}},\bar{\mathbf{z}}) = \left(\mathbf{x}+\bar{\mathbf{x}},\ \mathbf{y}+\bar{\mathbf{y}},\ \mathbf{z}+\bar{\mathbf{z}}+\frac{\mathbf{x}\bar{\mathbf{y}}}{\mathbf{2}}-\frac{\bar{\mathbf{x}}\mathbf{y}}{\mathbf{2}}\right).$$

Remark that the mapping

$$\mathbb{R}^{\mathbf{3}} \to \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle|\ \mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R} \right\} : (\mathbf{x},\mathbf{y},\mathbf{z}) \mapsto \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism between $(\mathbb{R}^3, *)$ and a subgroup of $\mathrm{GL}(3,\mathbb{R})$.
For every $\tau \neq 0$: left-invariant Riemannian metric on $(\mathbb{R}^3, *)$
$g = dx^2 + dy^2 + 4\tau^2 \left(dz + \frac{y\,dx - x\,dy}{2}\right)^2$. After the change of coordinates $(x, y, 2\tau z) \mapsto (x, y, z)$, $g = dx^2 + dy^2 + (dz + \tau(y\,dx - x\,dy))^2$

$$\mathrm{Nil}_3 = (\mathbb{R}^3, *) \text{ with } \mathbf{g}.$$

Some authors: only if $\tau = \frac{1}{2}$.

CAS in Heisenberg group Nil_3

The following vector fields form a left-invariant orthonormal frame on Nil_3:

$$e_1 = \partial_x - \tau y \partial_z, \quad e_2 = \partial_y + \tau x \partial_z, \quad e_3 = \partial_z.$$

The geometry of Nil_3 can be described in terms of this frame.

The Killing vector field $\boxed{e_3}$ plays an important role in the geometry of Nil_3.

Definition

We say that a surface in the Heisenberg group Nil_3 is a *constant angle surface* if the angle θ between the unit normal and the direction e_3 is the same at every point.

- cannot have $\theta = 0$ - contradiction with: $[e_1, e_2] = 2\tau e_3$, $[e_2, e_3] = 0$ $[e_3, e_1] = 0$, since $\tau \neq 0$.

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CAS in Heisenberg group Nil_3

Theorem (Fastenakels, Munteanu, Van der Veken)

Let M be a constant angle surface in the Heisenberg group Nil_3. Then M is isometric to an open part of one of the following types of surfaces:

(i) a Hopf-cylinder,

(ii) a surface given by

$$r(u,v) = \Big(\tfrac{1}{2\tau} \tan\theta \sin u + f_1(v),\ -\tfrac{1}{2\tau} \tan\theta \cos u + f_2(v),$$
$$-\tfrac{1}{4\tau} \tan^2\theta\, u - \tfrac{1}{2} \tan\theta \cos u f_1(v) - \tfrac{1}{2} \tan\theta \sin u f_2(v) - \tau f_3(v) \Big)$$

with $(f_1')^2 + (f_2')^2 = \sin^2\theta$ and $f_3'(v) = f_1'(v) f_2(v) - f_1(v) f_2'(v)$.

J. Fastenakels, M.I. Munteanu, J. Van der Veken, *Constant Angle Surfaces in the Heisenberg group*, Acta Math. Sinica(Engl. Ser.), 27 (2011) 4, 747–756.

CAS in Solvable Lie groups

... to be continued...

Thank you for attention !