Least Area Planes in $\mathbf{H}^3$ are Properly Embedded

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Basic Definitions

- Let Σ be a surface in a Riemannian manifold M. We call Σ a **minimal surface** if the mean curvature is 0 everywhere.
- A **least area disk** is a disk which has the smallest area among the disks with the same boundary.

  A **least area plane** is a plane such that any compact subdisk in the plane is a least area disk.
- A compact, orientable surface with boundary is called **absolutely area minimizing surface** if it has the smallest area among all orientable surfaces (with no topological restriction) with the same boundary.

  A noncompact, orientable surface is called **absolutely area minimizing surface** if any compact subsurface is an absolutely area minimizing surface.
- Any least area disk, and area minimizing surface is automatically a minimal surface. The main difference between least area disk and area minimizing surface is that there is no topological restriction on the surface.

Asymptotic Plateau Problem

Plateau Problem

Let Γ be a simple closed curve in M. Then, is there a least area disk (or absolutely area minimizing surface) Σ in M with $\partial\Sigma = \Gamma$?

Asymptotic Plateau Problem

Let Γ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Then, is there a least area plane (or absolutely area minimizing surface) Σ in $\mathbf{H}^3$ with $\partial_\infty\Sigma = \Gamma$?

- **Existence 1:** [Anderson, 83] For any $\Gamma \subset S^2_\infty(\mathbf{H}^3)$, there exists a least area plane Σ in $\mathbf{H}^3$ with $\partial_\infty\Sigma = \Gamma$.
- **Existence 2:** [Anderson, 82] For any codimension-1 hypersurface Γ in $S^n_\infty(\mathbf{H}^{n+1})$, there exists an absolutely area minimizing hypersurface Σ in $\mathbf{H}^{n+1}$ with $\partial_\infty\Sigma = \Gamma$.
- **Regularity:** [Hardt-Lin 1987, Tonegawa 1996] Let Γ be a C^1 simple closed curve in $S^2_\infty(\mathbf{H}^3)$, and let Σ be a minimal surface in $\mathbf{H}^3$ with $\partial_\infty\Sigma = \Gamma$. Then, $\Sigma \cup \Gamma$ is a C^1 manifold with boundary in $\overline{\mathbf{H}^3}$.

Asymptotic Plateau Problem

- **Number of Solutions:**

  Let Γ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$.

- [Anderson, 83] If Γ bounds a **convex** domain in $S^2_\infty(\mathbf{H}^3)$, then Γ bounds a unique absolutely area minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$.

- [Hardt-Lin, 87] If Γ bounds a **star-shaped** domain in $S^2_\infty(\mathbf{H}^3)$, then Γ bounds a unique absolutely area minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$.

- [C-, 05] For a **generic** simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$, Γ bounds a unique absolutely area minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$.
  The same is true for least area planes, too.

- These results are also valid for general dimensions where Σ is the absolutely area minimizing hypersurface.

More Definitions

- An immersed surface S in $\mathbf{H}^3$ is **proper** if the preimage of any compact subset of $\mathbf{H}^3$ is compact in the surface S.

  If an embedded surface S in $\mathbf{H}^3$ is proper, we will call S as **properly embedded**.

- **Some examples**

- Let A be a subset of $S^2_\infty(\mathbf{H}^3)$. Then the **convex hull** of A, $CH(A)$, is the smallest closed convex subset of $\mathbf{H}^3$ which is asymptotic to A.

  Equivalently, $CH(A)$ can be defined as the intersection of all supporting closed half-spaces of $\mathbf{H}^3$.

Basic Facts

Convex Hull Property

Let Σ be a minimal surface in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$, then $\Sigma \subset CH(\Gamma)$.

Maximum Principle

If Σ and Σ' are two minimal surfaces in M, then they cannot "touch" each other.

Meeks-Yau Exchange Roundoff Trick

If Σ and Σ' are two least area disks, then $\Sigma \cap \Sigma'$ cannot contain a closed curve.

In asymptotic Plateau problem setting, this implies the following:

Let Γ_1, Γ_2 be two **disjoint** simple closed curves in $S^2_\infty(\mathbf{H}^3)$. If Σ_1 and Σ_2 are two least area planes in $\mathbf{H}^3$ with $\partial_\infty \Sigma_i = \Gamma_i$, then Σ_1 and Σ_2 are **disjoint**, too.

The Main Result

Question

Let Γ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Let Σ be a least area plane in $\mathbf{H}^3$ with $\partial_\infty\Sigma = \Gamma$. Must such a least area plane Σ be properly embedded in $\mathbf{H}^3$?

Main Result

Let Γ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ containing at least one C^1 point, and let Σ be a least area plane in $\mathbf{H}^3$ with $\partial_\infty\Sigma = \Gamma$. Then Σ is properly embedded in $\mathbf{H}^3$.

Analogous Question in $\mathbf{R}^3$:

- ◇ [Colding-Minicozzi, 08] Any complete, embedded minimal surface of finite topology in $\mathbf{R}^3$ must be **properly embedded.**
- ◇ They used **chord-arc bound** for the proof.
- ◇ This is known as the **Calabi-Yau Conjecture for Minimal Surfaces**.

Proof of the Main Result

- From now on, Γ represents a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ containing at least one C^1 point, and

  Σ represents a least area plane in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$.
- **Lemma 1:** For a generic $r > 0$, $B_r(0) \cap \Sigma$ consists of disjoint least area disks.
- **Lemma 2:** If Σ is not properly embedded, then $\exists r_0 > 0$ such that for any generic $r > r_0$, $B_r(0) \cap \Sigma$ consists of infinitely many disjoint least area disks.
- **Definition:** Separating Disks vs. Nonseparating Disks

Key Lemma

Key Lemma

Let $D_r \subset \Sigma \cap B_r(0)$ be a nonseparating disk.
Then $d(0, D_r) > F(r)$ where F is monotone increasing function $F : (C, \infty) \to (C', \infty)$, and $F(r) \nearrow \infty$ as $r \nearrow \infty$.

- If Σ is nonproper, then intuitively Σ should visit the compact part many times "unnecessarily". This lemma prohibits this!
- The Proof of the Key Lemma:

  Step 1. Existence of Least Area Annulus

  Step 2. Nonseparating Disks Stays near Boundary

Proof of the Main Theorem

- **Lemma:** If Σ is nonproper, then for any $R > 0$, there exists $R' > R$ such that $B_{R'}(0) \cap \Sigma$ consists of infinitely many **separating** disks.

- **Main Theorem:** Σ is properly embedded in $\mathbf{H}^3$.

Final Remarks

Conjecture

Main result is true without the smooth point condition.

- However;

  ◇ If we relax the condition being **least area** to being **minimal surface**, there are examples of nonproperly embedded, complete, minimal planes in $\mathbf{H}^3$ [C-, 10].

  ◇ If we allow **more than one component** for $\partial_\infty \Sigma$, there are examples of nonproperly embedded, complete least area planes in $\mathbf{H}^3$ [Meeks-Tinaglia, 10].

- In other words, CYC in $\mathbf{H}^3$ is not true.

We will talk more about the examples above in the next talk.

References

1. M. Anderson, *Complete minimal varieties in hyperbolic space*, Invent. Math. **69** (1982) 477–494.
2. M. Anderson, *Complete minimal hypersurfaces in hyp. n-manifolds*, Comment. Math. Helv. **58** (1983) 264–290.
3. T.H. Colding and W.P. Minicozzi, *The Calabi-Yau conjectures for embedded surfaces*, Ann. of Math. (2) **167** (2008) 211–243.
4. B. Coskunuzer, *Least Area Planes in Hyperbolic 3-Space are Properly Embedded*, Indiana Univ. Math. J. **58** (2009) 381-392.
5. B. Coskunuzer, *Generic Uniqueness of Least Area Planes in Hyp. Space*, Geom. & Topology **10** (2006) 401–412.
6. B. Coskunuzer, *On the Number of Solutions to Asymptotic Plateau Problem*, math/0505593.
7. B. Coskunuzer, *Non-properly Embedded Minimal Planes in Hyp. 3-Space* , Comm. Contemp. Math. 13 (2011) 727-739.
8. D. Gabai, *On the geometric and topological rigidity of hyp.* 3*-manifolds*, J. Amer. Math. Soc. **10** (1997) 37–74.
9. R. Hardt and F.H. Lin, *Regularity at infinity for absolutely area minimizing hypersurfaces in hyperbolic space*, Invent. Math. **88** (1987) 217–224.
10. W. Meeks and S.T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. **112** (1980) 441–484.
11. W.H. Meeks, and G. Tinaglia, *Properness results for constant mean curvature surfaces*, preprint.
12. Y. Tonegawa, *Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space*, Math. Z. **221** (1996) 591–615.