A counterexample to Calabi-Yau Conjecture in $\mathbf{H}^3$

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21 October 2011

Basic Definitions

- Let Σ be a surface in a Riemannian manifold M. We call Σ a **minimal surface** if the mean curvature is 0 everywhere.

- A **least area disk** is a disk which has the smallest area among the disks with the same boundary.

  A **least area plane** is a plane such that any compact subdisk in the plane is a least area disk.

- A compact, orientable surface with boundary is called **absolutely area minimizing surface** if it has the smallest area among all orientable surfaces (with no topological restriction) with the same boundary.

  A noncompact, orientable surface is called **absolutely area minimizing surface** if any compact subsurface is an absolutely area minimizing surface.

- Any least area disk, and area minimizing surface is automatically a minimal surface. The main difference between least area disk and area minimizing surface is that there is no topological restriction on the surface.

Calabi-Yau Conjecture in $\mathbf{R}^3$

Calabi-Yau Conjecture

A complete, embedded minimal surface in $\mathbf{R}^3$ is proper.

- **Finite Topology case:** [Colding-Minicozzi] The conjecture is true for minimal surfaces with finite genus & finite number of ends in $\mathbf{R}^3$.
- **Finite Genus & Countable ends:** [Meeks-Perez-Ros] The conjecture is true for minimal surfaces with finite genus & countable number of ends in $\mathbf{R}^3$.
- **Positive Injectivity Radius case:** [Meeks-Rosenberg] The conjecture is true for minimal surfaces with positive injectivity radius in $\mathbf{R}^3$.

  (Implies finite topology case [Colding-Minicozzi])
- **Finite Genus case:** finite genus & uncountable number of ends case is still open.

Generalizations of CYC

Question

Is it possible to generalize CYC to simply connected, nonpositive curvature spaces?

- Why simply connected? Why nonpositive curvature?
- **Supporting Result:** [Meeks-Rosenberg]
  Let N be a Riemannian 3-manifold of nonpositive sectional curvature.
  If M is a complete embedded minimal surface of finite topology in N, then $\overline{M}$ has the structure of a **minimal lamination**.
- Note that the result above is the key lemma for their proof of CYC for finite genus case in $\mathbf{R}^3$.
- **Question:** Is CYC true for simply connected, nonpositive curvature spaces?

CYC vs. $\mathbf{H}^3$

- **Result 1:** [Tonegawa] Let M be a complete **minimal surface** in $\mathbf{H}^3$ where $\partial_\infty M = \Gamma$ is a C^1 simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Then, M is properly embedded.

- **Result 2:** [C–] Let Σ be a complete **least area plane** in $\mathbf{H}^3$ where $\partial_\infty M = \Gamma$ is a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ with one smooth point. Then, Σ is properly embedded.

Question

Is CYC true for $\mathbf{H}^3$?

- **NO**. In this talk, we will construct a counterexample to CYC in $\mathbf{H}^3$?

- [Meeks-Tinaglia] have constructed another counterexample recently.

A counterexample to CYC

Main Theorem

There exists a nonproperly embedded minimal plane in $\mathbf{H}^3$.

- The example:
  - ◇ Take sequence of circles C_n in $S^2_\infty(\mathbf{H}^3)$ limiting on equator.
  - ◇ Each C_n bounds a geodesic plane P_n in $\mathbf{H}^3$
  - ◇ Connect P_n and P_{n+1} with a bridge at infinity
  - ◇ Resulting plane Σ is nonproperly embedded.
- The construction is not trivial since we do not have *the bridge principle at infinity*.

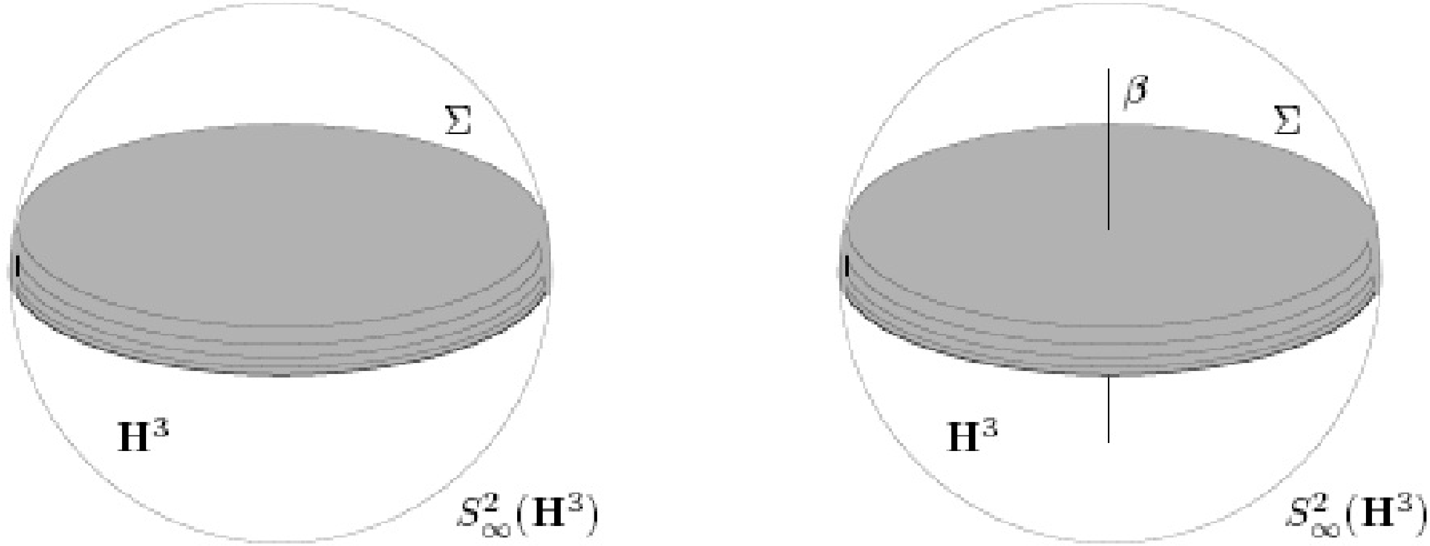

FIGURE 4. In the figure left, we see the minimal plane Σ with $\partial_\infty \Sigma = \Gamma$. In the figure right, we see the line segment β which is transverse to Σ.

The Construction

Main Ingredients

Lemma (Meeks-Yau)

Let Ω be a compact, mean convex 3-manifold, and $\alpha \subset \partial\Omega$ be a nullhomotopic simple closed curve.
Then, there exists an **embedded least area disk** *$D \subset M$ with $\partial D = \Gamma$.*

Lemma (Gabai)

Let $\{E_n\}$ be a sequence of embedded least area disks in $\mathbf{H}^3$ where $\partial E_n \to \infty$. Then after passing to a subsequence $\{E_{n_j}\}$ converges to a (possibly empty) **lamination** *σ by least area planes in $\mathbf{H}^3$.*

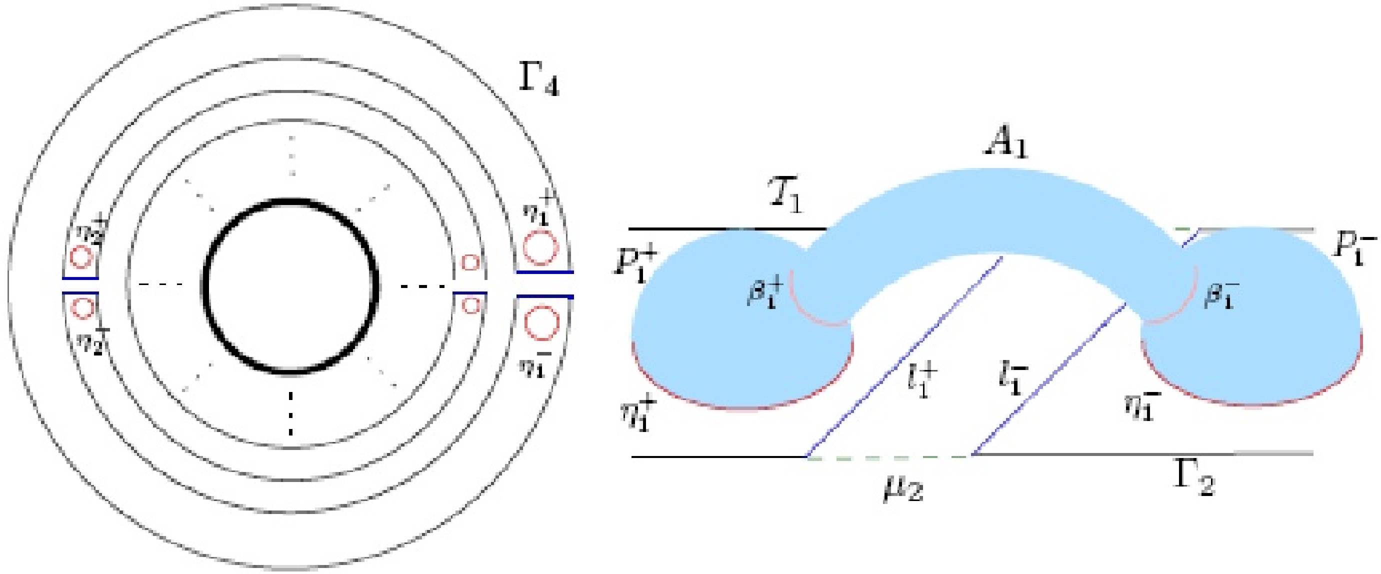

FIGURE 2. In the figure left, Γ_4 is constructed by using the circles C_1, C_2, C_3, C_4 in $S^2_\infty(\mathbf{H}^3)$ by connecting each other with bridges (blue line segments $l_i^\pm$). The red circles represents $\eta_i^\pm$. In the figure right, the tunnel $\mathcal{T}_1$ is shown.

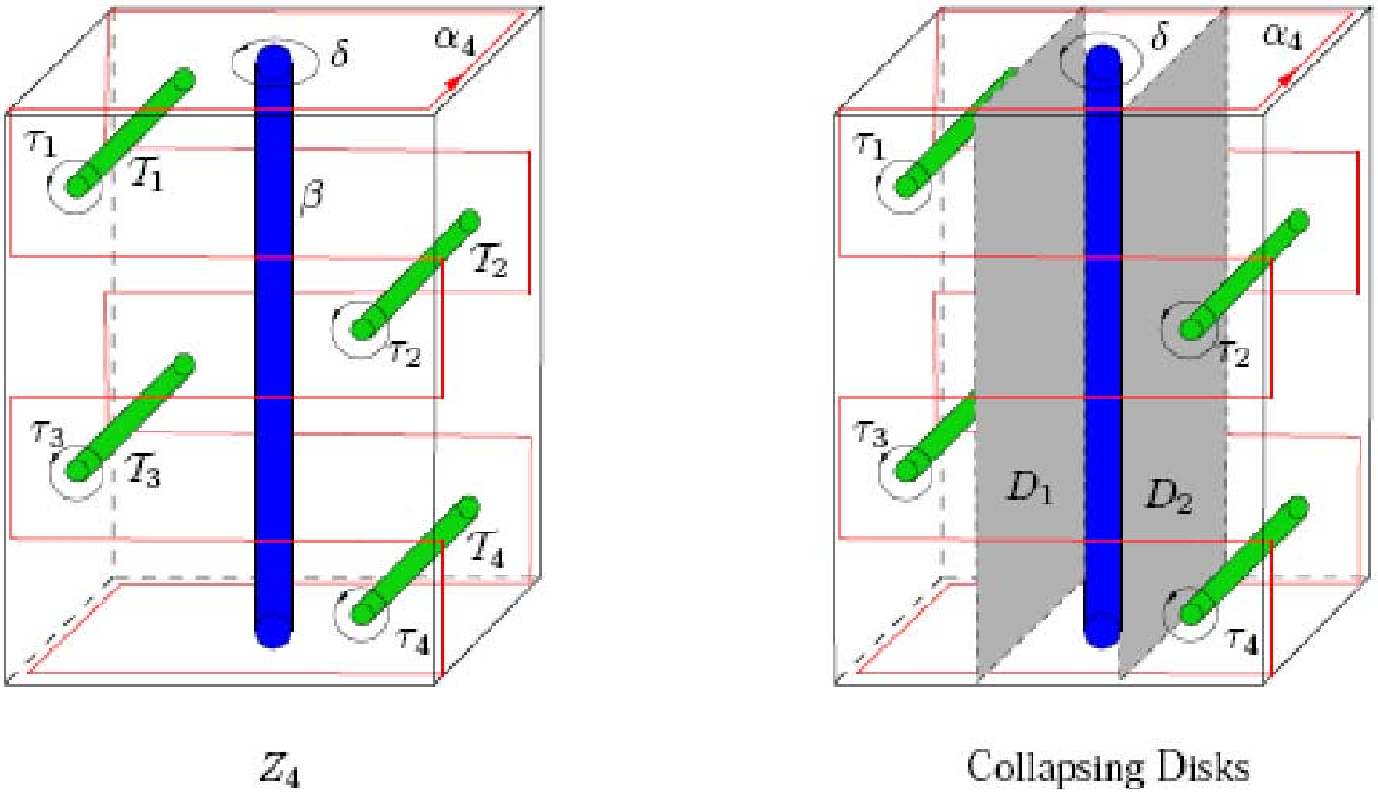

FIGURE 3. The box corresponds to Z_4. Green tubes are the tunnels, and blue tube is the curve β. The red curve is α_4. In the right, the grey rectangles are the collapsing disks.

Final Remarks

[Meeks-Rosenberg] technique for $\mathbf{H}^3$

What goes wrong in [Meeks-Rosenberg] technique while Key Lemma is valid?

[Meeks-Rosenberg] Theorem 3 (Key Lemma)

Let N be a 3-manifold with nonpositive curvature. Let M be a complete embedded minimal surface with finite topology in N. Then, $\overline{M}$ has the structure of a minimal lamination.

[Meeks-Rosenbeg] Theorem 4

M has bounded curvature near a limit leaf L in $\overline{M}$.

[Meeks-Rosenberg2] Lemma 1.3

If $N = \mathbf{R}^3$, then M has unbounded curvature near the limit leaf L.

This proves CYC in $\mathbf{R}^3$ for finite topology case.

Our example shows that the last statement is not valid for $\mathbf{H}^3$.

Final Remarks

The Bridge Principle at Infinity

- Why can't we simply use the bridge principle in our construction?
- Is Bridge Principle at Infinity True?
  - ◇ Least Area case: *Probably Not*
  - ◇ Minimal case: *Probably Yes*

Final Remarks

[Meeks-Tinaglia] example

- Σ is a least area plane with $\partial_\infty \Sigma = C_1^\pm \cup C_2^\pm$ where $C_1^\pm$ and $C_2^\pm$ are two pairs of round circles in $S^2_\infty(\mathbf{H}^3)$.
- Σ is an infinite strip which spirals into two annuli A_1 and A_2 with $\partial_\infty A_i = C_i^+ \cup C_i^-$.
- [Meeks-Tinaglia] example generalizes to any CMC surface with mean curvature $H \in [0, 1)$.

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