Nonproper Minimal Surfaces with Arbitrary Topology in $\mathbf{H}^3$

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Basic Definitions

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- Any least area disk, and area minimizing surface is automatically a minimal surface. The main difference between least area disk and area minimizing surface is that there is no topological restriction on the surface.

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- **Finite Genus case:** Finite genus & uncountable number of ends case is still open.
- **Constant Mean Curvature case:** [Meeks-Tinaglia] The conjecture is true for H-surfaces in $\mathbf{R}^3$.

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- There exists a complete, nonproper, minimal plane in $\mathbf{H}^3$. [C–2011]

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Question

Are there other complete nonproper, minimal surfaces in $\mathbf{H}^3$?

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Question

What type of surfaces can be **nonproperly** embedded in $\mathbf{H}^3$ as a complete minimal surface?

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Any open, orientable surface S can be **nonproperly** embedded in $\mathbf{H}^3$ as a complete minimal surface.

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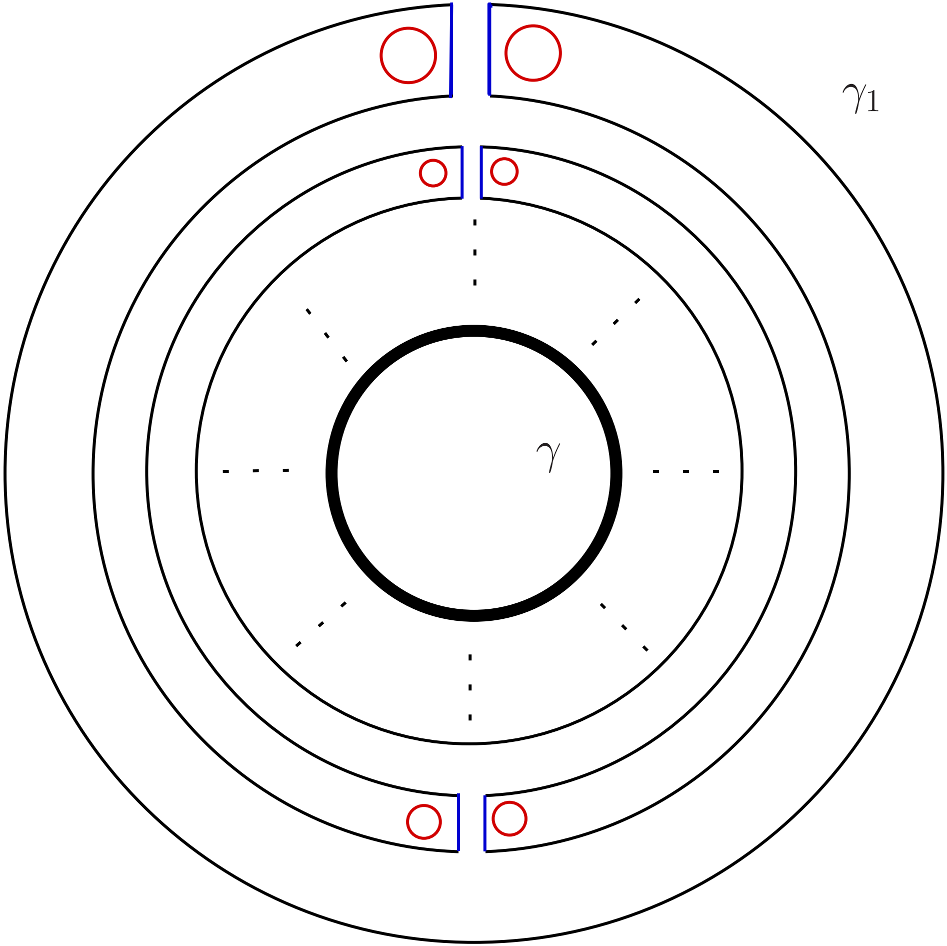
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γ

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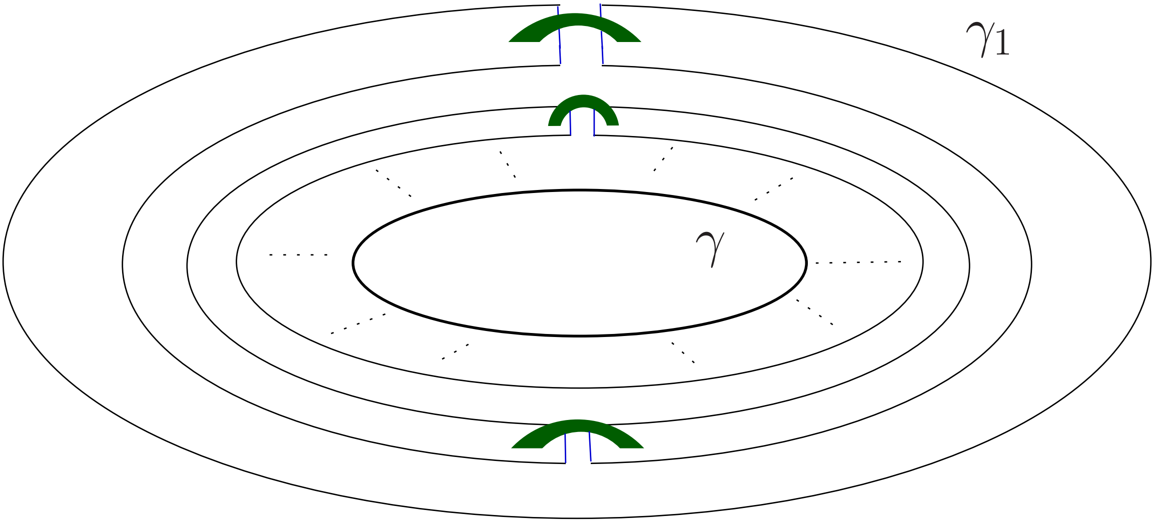
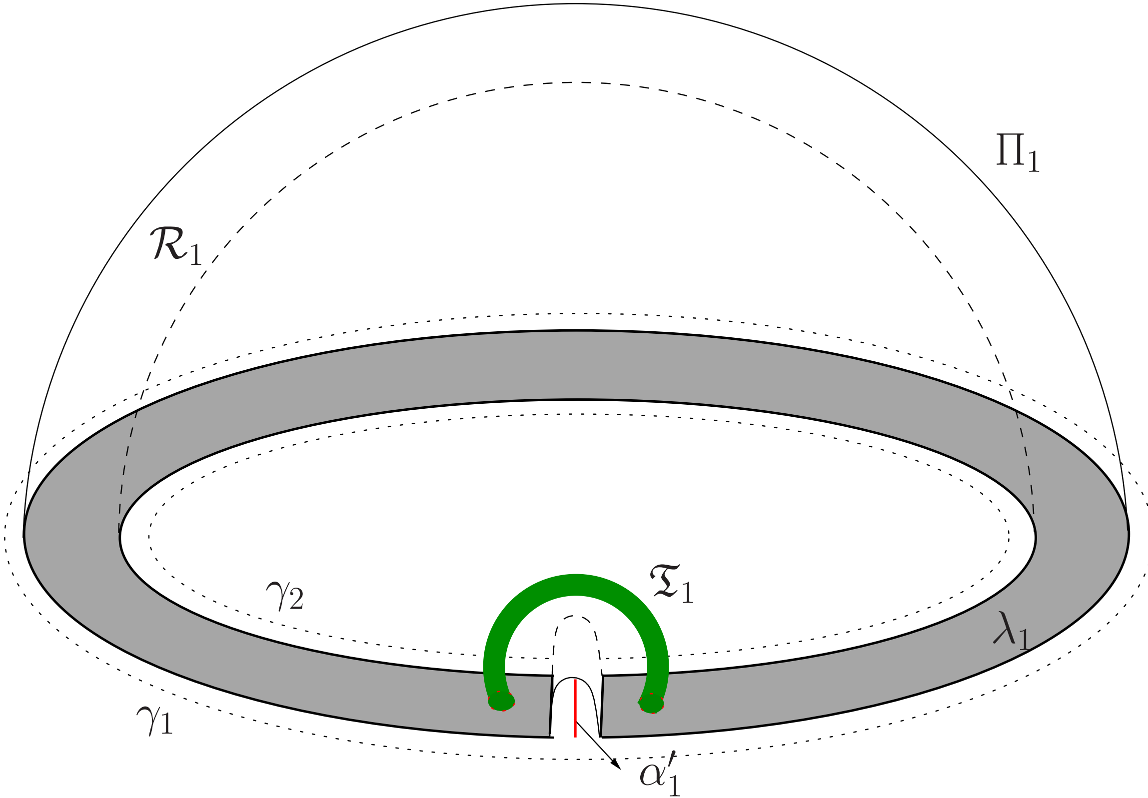
- The construction is not trivial since we do not have *the bridge principle at infinity* in $\mathbf{H}^3$ for stable minimal surfaces.


γ_1
γ



Π_1
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  i.e. $S = \bigcup_{n=1}^{\infty} S_n$ where $S_1 \subset S_2 \subset ... \subset S_n \subset ..$
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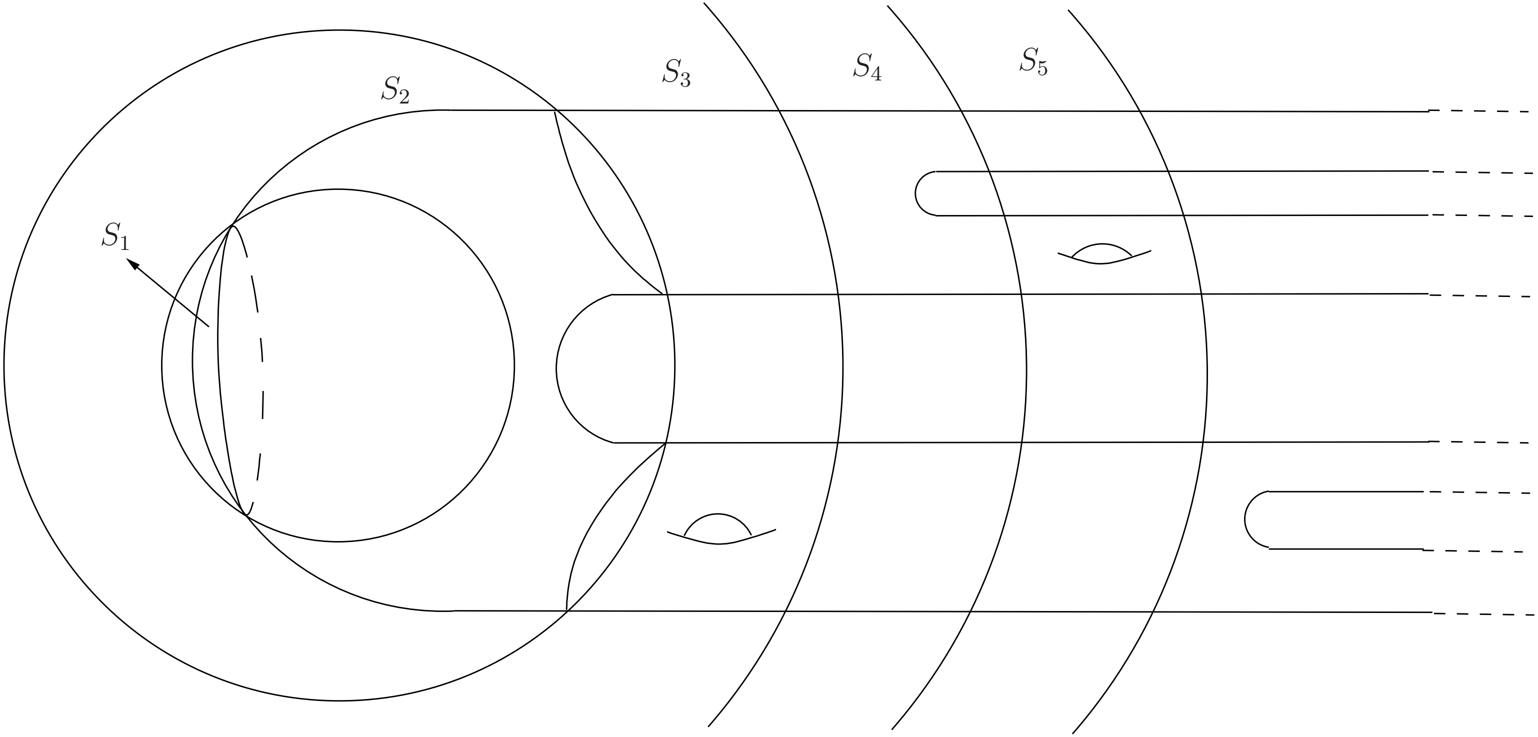
S_1
S_2
S_3
S_4
S_5

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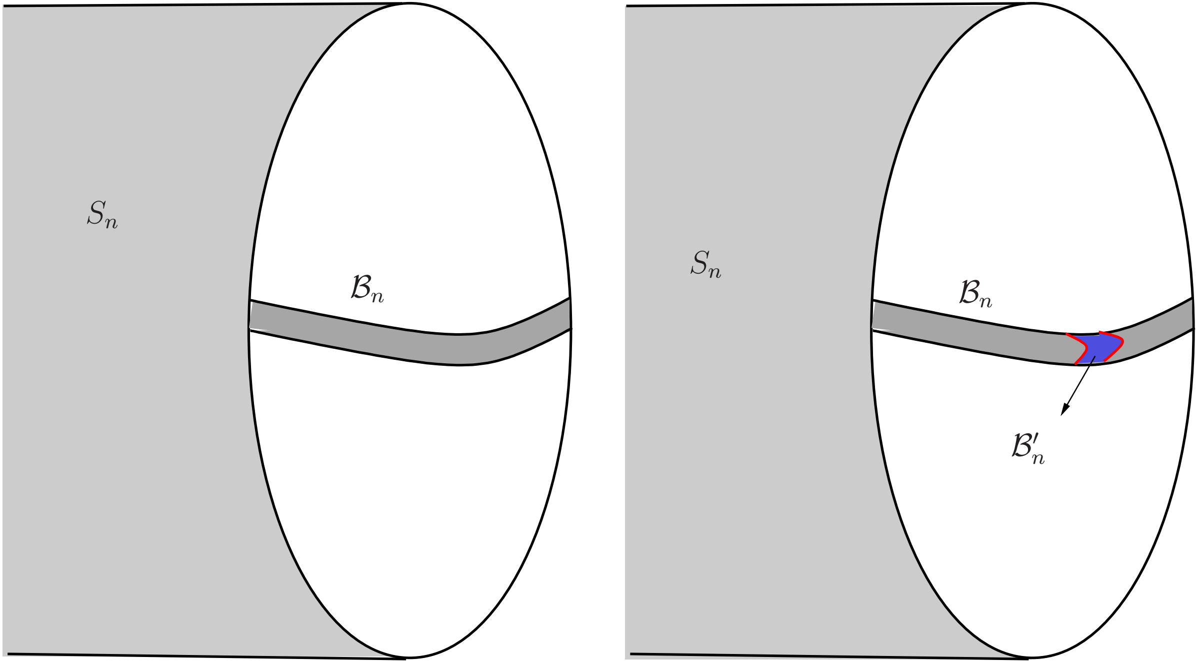
S_n
$\mathcal{B}_n$
S_n
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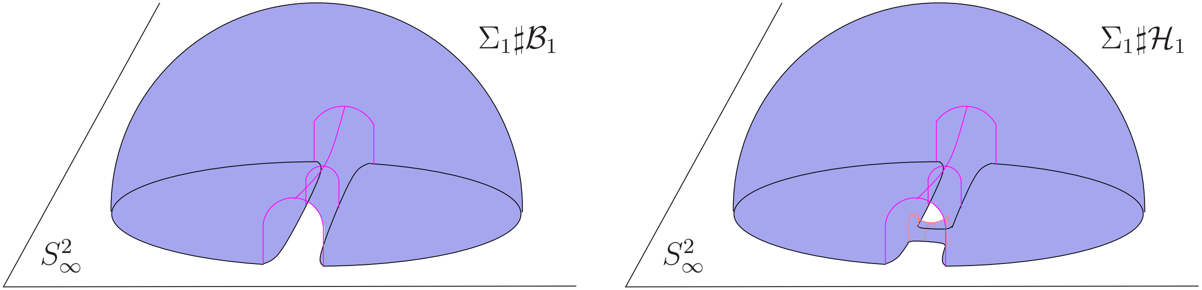
$\mathcal{B}'_n$



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S^2_∞
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NEED

Mean Convex Subspaces X_n in $\mathbf{H}^3$ where $T_{2n-1} \cup P_n$ is uniquely minimizing in X_n.

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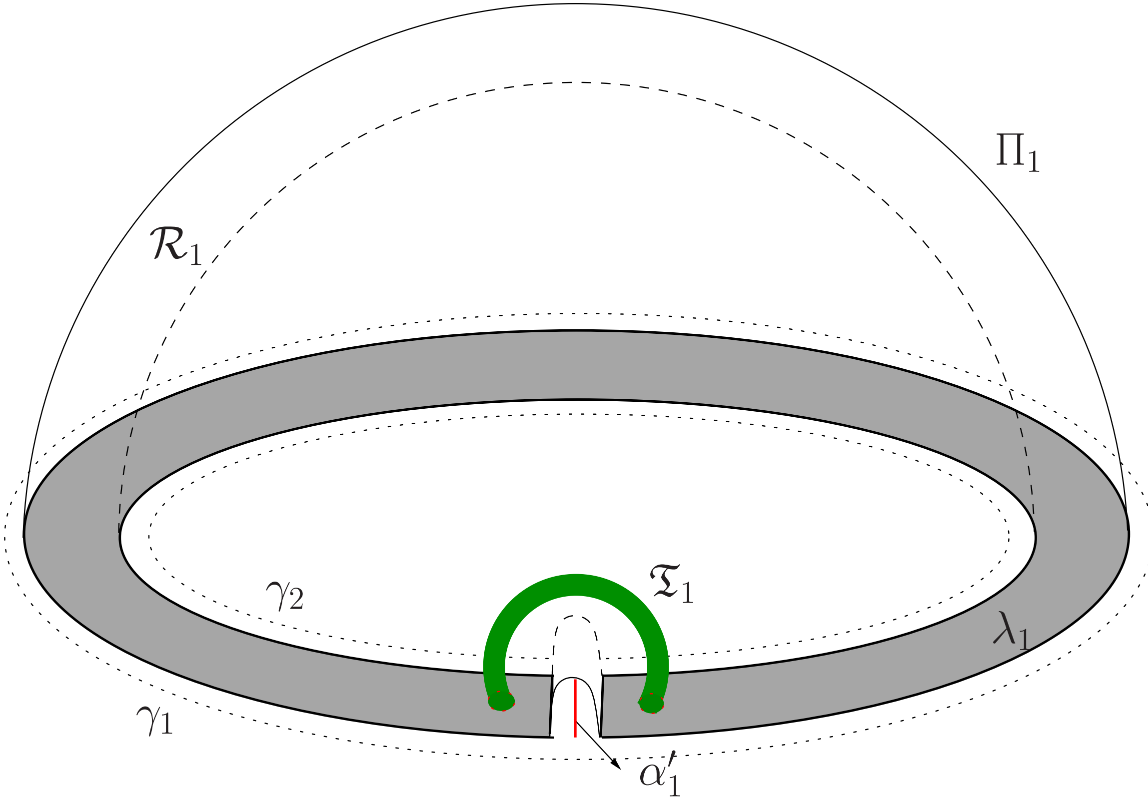

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- Σ is nonproper as $\overline{\Sigma} \supset \overline{\Sigma_1} \supset P_\infty$.

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$\diamond$ [Meeks-Tinaglia] For $H \geq 1$, Calabi-Yau Conjecture is true for H-surfaces in $\mathbf{H}^3$.

References

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