Complete proper holomorphic embeddings of strictly pseudoconvex domains into balls

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Abstract

We present a construction of a complete proper holomorphic embedding from any strictly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ into the unit ball of $\mathbb{C}^N$, for N large enough.

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  - Jones (1979) constructed complete bounded holomorphic immersed discs in $\mathbb{C}^2$ and embedded discs in $\mathbb{C}^3$.
  - Martín, Umehara, Yamada (2009) extended Jones's result to complete bounded complex curves in $\mathbb{C}^2$ with arbitrary finite genus and finitely many ends.
  - Alarcón, López (2013) constructed complete bounded immersed holomorphic curves in $\mathbb{C}^2$ with arbitrary topology; they also constructed complete properly embedded complex curves in any convex domain in $\mathbb{C}^2$ with no control of the topology.
  - Alarcón, Forstnerič (2013) constructed a complete proper holomorphic immersion from any bordered Riemann surface into the unit ball in $\mathbb{C}^2$, and a complete proper holomorphic embedding into the unit ball in $\mathbb{C}^m$, $m \geq 3$. They asked if there is a complete proper holomorphic immersion/embedding from the unit ball in $\mathbb{C}^n$ into the unit ball of a higher dimensional Euclidean space.

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- Globevnik (2014,2015) gave a positive answer for higher dimensional submanifolds: for any n, m, $1 \leq n < m$, there is a complete closed n-dimensional complex submanifold in any pseudoconvex domain of $\mathbb{C}^m$. There is no control on the topology of the submanifolds.

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- Alarcón, Globevnik, López (2015) constructed complete bounded embedded complex hypersurfaces in $\mathbb{C}^{n+1}$ with some control of the topology, in particular, they constructed complete properly embedded complex curves in the unit ball of $\mathbb{C}^2$ with arbitrary finite topology.

Main theorem

Theorem

Let D be a bounded strictly convex domain with $\mathcal{C}^2$-boundary in $\mathbb{C}^n$. There exists a positive integer s with the following property. For any positive integer p and for any continuous map $h\colon \overline{D} \to \mathbb{B}_p$ such that $h|_D$ is an injective holomorphic immersion, there exists a holomorphic map $f\colon D \to \mathbb{C}^{2s}$, such that the map $(f,h)\colon D \to \mathbb{B}_{2s+p}$ is a complete proper holomorphic embedding.

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By Fornaess' embedding theorem any bounded strictly pseudoconvex domain with $\mathcal{C}^2$-boundary embeds properly holomorphically into a strictly convex domain in Euclidean space.

Corollary

Let D be a bounded strictly pseudoconvex domain with $\mathcal{C}^2$-boundary in $\mathbb{C}^n$. For N large enough there exists a complete proper holomorphic embedding $F\colon D \to \mathbb{B}_N$.

About the method of the proof

We use holomorphic peak functions, the idea which goes back to Hakim and Sibony, and Løw to solve the inner function problem:

Theorem (Aleksandrov, Hakim, Sibony, Løw (1982))

There is a nonconstant bounded holomorphic function $u\colon \mathbb{B}_n \to \mathbb{C}$ such that $|u^*(z)| = 1$ a.e. on $b\mathbb{B}_n$ (inner function).

This result was unexpected among experts, partly because of the knowledge that such functions would have to exhibit extremely pathological boundary behavior.

Aleksandrov actually solved the inner function problem 6 weeks before Løw, by a different method.

About the method of the proof

The method of Hakim, Sibony and Løw soon afterwards gave the following

Theorem (Forstnerič, Løw (1985))

Let D be a bounded strictly pseudoconvex domain with $\mathcal{C}^2$-boundary in $\mathbb{C}^n$. For N large enough there exists a proper holomorphic embedding $F\colon D \to \mathbb{P}_N$ and a proper holomorphic embedding $F\colon D \to \mathbb{B}_N$. Moreover, the later embedding can be made continuous up to the boundary.

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More precise results with respect to the codimension were obtained later by Hakim, Stensønes, Dor:

Theorem (Stensønes (1989), Hakim (1990), Dor (1990))

Let D be a bounded strictly pseudoconvex domain with $\mathcal{C}^\infty$-boundary in $\mathbb{C}^n$. There exists a proper holomorphic map $F\colon D \to \mathbb{P}_{n+1}$ and a proper holomorphic map $F\colon D \to \mathbb{B}_{n+1}$ that extends continuously up to the boundary.

Auxiliary lemma

Let D is a bounded strictly convex domain with $\mathcal{C}^2$-boundary in $\mathbb{C}^n$. Let bD denote its boundary and $\nu(w)$ the outward unit normal to bD at the point $w \in bD$.
For $a \in \mathbb{C}^n$ and $r > 0$ let $\mathbb{B}(a, r)$ denote the open ball of radius r centered at a in $\mathbb{C}^n$.

Lemma

There are constants α_1, α_2, $r_1 > 0$ such that the following hold:

$$\begin{aligned} \Re\langle w - z, \nu(w)\rangle &\geq \alpha_1 \|z - w\|^2 \quad \forall w \in bD, z \in \overline{D}, \mathrm{dist}(z, bD) < r_1, \\ \Re\langle w - z, \nu(w)\rangle &\leq \alpha_2 \|z - w\|^2 \quad \forall z, w \in bD. \end{aligned}$$

Auxiliary covering lemma

Lemma

For every $\lambda > 1$ there exists an integer $s > 0$ with the following property: For each $r > 0$ there are s families of balls $\mathcal{F}_1, \ldots, \mathcal{F}_s$,

$$\mathcal{F}_i = \{\mathbb{B}(z_{i,j}, \lambda r) : 1 \leq j \leq N_i\},$$

with centers $z_{i,j} \in bD$, such that the balls in each family are pairwise disjoint, and

$$bD \subset \bigcup_{i=1}^{s} \bigcup_{j=1}^{N_i} \mathbb{B}(z_{i,j}, r).$$

Peak functions

Fix α_1 and α_2 from the first Lemma and let $\lambda = 4\sqrt{\frac{\alpha_2}{\alpha_1}}$.
We get an integer $s > 0$ satisfying the properties in the covering Lemma: for any $r > 0$ we have s families of balls $\mathcal{F}_1, \ldots, \mathcal{F}_s$, $\mathcal{F}_i = \{\mathbb{B}(z_{i,j}, \lambda r) : 1 \leq j \leq N_i\}$, $z_{i,j} \in bD$, such that the balls in each $\mathcal{F}_i$ are pairwise disjoint and $bD \subset \bigcup_{i=1}^{s} \bigcup_{j=1}^{N_i} \mathbb{B}(z_{i,j}, r)$.

For each $1 \leq i \leq s$ and $1 \leq j \leq N_i$ we define $z_{i+s,j} = z_{i,j}$ and $\mathcal{F}_{i+s} = \mathcal{F}_i$.

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For each $1 \le i \le s$ and $1 \le j \le N_i$ we define $z_{i+s,j} = z_{i,j}$ and $\mathcal{F}_{i+s} = \mathcal{F}_i$.
Further, for $m > 0$, $1 \le i \le 2s$ and $1 \le j \le N_i$ we define

$$\phi_{i,j}(z) = e^{-m\langle z_{i,j} - z, \nu(z_{i,j})\rangle}, \ z \in \overline{D}.$$

We get the following estimates

$$\begin{aligned} |\phi_{i,j}(z)| &\le e^{-\alpha_1 m\|z - z_{i,j}\|^2} & \forall z \in \overline{D}, \mathrm{dist}(z, bD) < r_1, \\ |\phi_{i,j}(z)| &\ge e^{-\alpha_2 m\|z - z_{i,j}\|^2} & \forall z \in bD. \end{aligned}$$

Properties of functions g_i

For given $|\beta_{i,j}| \leq 1$, let g_i be the entire function

$$g_i(z) = \sum_{j=1}^{N_i} \beta_{i,j}\phi_{i,j}(z), \qquad z \in \overline{D}.$$

Lemma

For each sufficiently small $\eta > 0$ there are $m, r > 0$, $0 < \lambda r < r_1$, such that for each i, $1 \leq i \leq 2s$, the following hold:

(a) *If a point $z \in bD$ lies in no ball in $\mathcal{F}_i$, then $|g_i(z)| < \eta$.*

(b) *If $z \in \overline{D} \cap \mathbb{B}(z_{i,j}, \lambda r)$ for some j, then $|g_i(z) - \beta_{i,j}\phi_{i,j}(z)| < \eta$.*

(c) *If $z \in bD \cap \mathbb{B}(z_{i,j}, r)$ for some j, then $|\phi_{i,j}(z)| \geq C\eta^{\frac{1}{16}}$, where the constant C is independent of r, m and η.*

(d) *If $z \in \overline{D} \cap b\mathbb{B}(z_{i,j}, \lambda r)$ for some j, then $|\phi_{i,j}(z)| < \eta^{\frac{2}{3}}$.*

Moreover, we can choose $r > 0$ arbitrarily small and make $m > 0$ as large as we want.

Lemma

There is $\epsilon_0 > 0$ such that the following holds: If we are given

(i) *numbers a and ϵ, $0 < \epsilon < \epsilon_0$, such that $a - \epsilon^{\frac{1}{2}} > \frac{1}{2}$ and $a + \epsilon < 1$,*

(ii) *a compact subset $K \subset D$,*

(iii) *a continuous map $f = (f_1, \ldots, f_{2s}) \colon \overline{D} \to \mathbb{C}^{2s}$, holomorphic in D, such that for the map $F = (f, h)$ we have $\|F(z)\| < a - \epsilon^{\frac{1}{2}}$ for each $z \in bD$,*

(iv) *a point $p \in D$ and a number $\sigma > 0$ such that $\mathrm{dist}_F(p, bD) > \sigma$, and*

(v) *a number $\delta > 0$,*

then there is an entire map $G = (g_1, \ldots, g_{2s}, 0, \ldots, 0) \colon \mathbb{C}^n \to \mathbb{C}^{2s+p}$:

(a) $\|(F+G)(z)\| \leq a + \epsilon$ *for all $z \in bD$,*

(b) *if $\|(F+G)(z)\| \leq a - \epsilon^{\frac{1}{7}}$ for some $z \in bD$, then $\|(F+G)(z)\| > \|F(z)\| + \epsilon^{\frac{2}{7}}$,*

(c) $\|G(z)\| < \delta$ *for all $z \in K$,*

(d) $\|G(z)\|^2 < 1 - \|F(z)\|$ *for all $z \in bD$,*

(e) $\mathrm{dist}_{F+G}(p, bD) > \sigma + E\epsilon^{\frac{5}{16}}$, *where E depends only on ϵ_0.*

Idea of the proof of the Lemma

We define the coefficients $\beta_{i,j}$ and $\beta_{i+s,j}$ as follows:

$$\begin{aligned} f_i(z_{i,j})\overline{\beta_{i,j}} + f_i(z_{i+s,j})\overline{\beta_{i+s,j}} &= 0, \\ |\beta_{i,j}|^2 + |\beta_{i+s,j}|^2 &= \frac{a^2 - \|F(z_{i,j})\|^2}{2s}. \end{aligned}$$

This implies that the vector $(\beta_{i,j}, \beta_{i+s,j})$ is perpendicular to the vector $(f_i(z_{i,j}), f_i(z_{i+s,j}))$ and $|\beta_{i,j}| < 1$, $|\beta_{i+s,j}| < 1$.
Then the entire map $G = (g_1, \ldots, g_{2s}, 0, \ldots, 0)$ has the properties (a)-(e), provided that the constant $m > 0$ is chosen large enough and $r > 0$ is chosen small enough.

Proof of the Theorem-1

We choose an increasing sequence $\{a_k\}_{k\geq 1}$ converging to 1, and a decreasing sequence $\{\epsilon_k\}_{k\geq 1}$ converging to 0, such that:

(i) $\max\{\sup_{bD} \|h\|, \frac{1}{2}\} < a_1 - \epsilon_1^{\frac{1}{2}}$,

(ii) $\sum_{k=1}^{\infty} \epsilon_k^{\frac{1}{2}} < \infty, \ \sum_{k=1}^{\infty} \epsilon_k^{\frac{5}{16}} = \infty$,

(iii) $a_k + \epsilon_k < a_{k+1} - \epsilon_{k+1}^{\frac{1}{2}}$ for all $k \geq 1$.

Let $F_0 = (0, \ldots, 0, h)$ and fix any $p \in D$. Since h is nonconstant we have $\mathrm{dist}_{F_0}(p, bD) > 0$. Using previous Lemma we construct inductively a sequence of entire maps $\{G_j \colon \mathbb{C}^n \to \mathbb{C}^{2s+p}\}$, a sequence of injective holomorphic immersions $F_k = F_0 + \sum_{j=1}^{k} G_j$, two increasing sequences of compact subsets $\{K_k\}_{k\geq 1}$, $\{L_k\}_{k\geq 1}$ of D such that

$$L_k \Subset \mathring{K}_k, \quad \text{and} \quad \bigcup_{k=1}^{\infty} K_k = \bigcup_{k=1}^{\infty} L_k = D,$$

a decreasing sequence $\{\delta_k\}_{k\geq 1}$ converging to 0, $0 < \delta_k < \epsilon_k$, such that:

Proof of the Theorem-2

(a) $\|F_{k-1}(z)\| \geq \min_{w \in bD} \|F_{k-1}(w)\| - \frac{1}{2^k}$ for each $z \in \overline{D} \setminus K_k$,

(b) $\|F_k(z)\| \leq a_k + \epsilon_k$ for each $z \in \overline{D}$,

(c) if $\|F_k(z)\| \leq a_k - \epsilon_k^{\frac{1}{7}}$ for some $z \in bD$, then
$\|F_k(z)\| > \|F_{k-1}(z)\| + \epsilon_k^{\frac{2}{7}}$,

(d) $\|G_k(z)\| < \frac{\delta_k}{2^k}$ for each $z \in K_k$,

(e) $\|G_k(z)\|^2 < 1 - \min_{w \in bD} \|F_k(w)\|$ for all $z \in \overline{D}$,

(f) $\mathrm{dist}_{F_{k-1}}(p, bL_k) > \frac{1}{2}\mathrm{dist}_{F_0}(p, bD) + E \sum_{j=1}^{k-1} \epsilon_j^{\frac{5}{16}}$,

(g) if $F\colon D \to \mathbb{C}^{2s+p}$ is holomorphic and $\|F(z) - F_{k-1}(z)\| < \delta_k$ for all $z \in K_k$, then $\mathrm{dist}_F(p, bL_k) > \mathrm{dist}_{F_{k-1}}(p, bL_k) - 1$.

Proof of the Theorem-3

Property (d) implies that the sequence F_k converges uniformly on compact sets in D to a holomorphic map $F\colon D \to \mathbb{C}^{2s+p}$ and we get the estimate

$$\begin{aligned} \|F_{k-1}(z) - F(z)\| &\leq \|F_{k-1}(z) - F_k(z)\| + \|F_k(z) - F_{k+1}(z)\| + \cdots \\ &\leq \frac{\delta_k}{2^k} + \frac{\delta_{k+1}}{2^{k+1}} + \cdots \leq \delta_k \quad \text{for every} \quad z \in K_k \end{aligned}$$

This implies together with (f) and (g) that

$$\mathrm{dist}_F(p, bL_k) > \frac{1}{2}\mathrm{dist}_{F_0}(p, bD) + E\sum_{j=1}^{k-1} \epsilon_j^{\frac{5}{16}} - 1.$$

By (ii) the series $\sum_j \epsilon_j^{\frac{5}{16}}$ diverges, which implies that the map F is complete. Property (b) and the maximum principle imply that $F(D) \subset \mathbb{B}_{2s+p}$. Since the map h is an injective immersion on D, $F_0 = (0, \ldots, 0, h)$ and all last p components of the maps G_k are zero for each k, all the maps F_k and the limit map F are injective immersions. The map F is proper since the series $\sum_j \epsilon_j^{\frac{2}{7}}$ is divergent by (ii).