Minimal Surfaces in the Heisenberg Space

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- J. M. Manzano, – : Height and Area Estimates for Constant Mean Curvature Graphs in Homogeneous Space.

arXiv: 1504.05239 [math.DG] (2015)

- –, R. Sa Earp, E. Toubiana: Minimal Graphs in Nil3: existence and non-existence result

arXiv:1508.01724 [math.DG] (2015)

Contents

- Nil(τ)
- Examples
- Existence
- Area Growth
- Height Growth
- An Open Problem

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- We are able to prove the analogous of some of our results in $\mathit{Nil}_3(\tau)$ also in $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{PSL_2(\mathbb{R})}$ (by direct computation with the suitable metric or by Daniel correspondence).

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- We are able to prove the analogous of some of our results in $\mathit{Nil}_3(\tau)$ also in $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{PSL_2}(\mathbb{R})$ (by direct computation with the suitable metric or by Daniel correspondence).
- $\mathit{Nil}_3(\tau)$ is also known as an $\mathbb{E}(\kappa, \tau)$ space, with $\kappa = 0$.

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A model for $Nil_3(\tau)$ is $\mathbb{R}^3$ endowed with the Riemannian metric

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- A global orthonormal frame is $E_1 = \partial_1 - \tau x_2 \partial_3$, $E_2 = \partial_2 + \tau x_1 \partial_3$, $E_3 = \partial_3$.

Isometries in $Nil_3(\tau)$

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$$\varphi_1(x_1, x_2, x_3) = (x_1 + c, x_2, x_3 + \tau c x_2)$$
$$\varphi_2(x_1, x_2, x_3) = (x_1, x_2 + c, x_3 - \tau c x_1)$$
$$\varphi_3(x_1, x_2, x_3) = (x_1, x_2, x_3 + c)$$
$$\varphi_4(x_1, x_2, x_3) = ((\cos\theta)x_1 - (\sin\theta)x_2, (\sin\theta)x_1 + (\cos\theta)x_2, x_3)$$
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- The curve $\varphi(\Gamma)$ is not contained in the x_1-x_2 plane in general.

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  The projection $\pi(\varphi(\Gamma))$ of such curve on the x_1-x_2 plane is obtained from the curve Γ by an isometry on the Euclidean x_1-x_2 plane.
- If Γ is convex, then $\pi(\varphi(\Gamma))$ is convex, for any isometry φ of $Nil_3(\tau)$.

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$$2H(u) := \mathrm{div}\left(\frac{Gu}{\sqrt{1 + \|Gu\|^2}}\right) = 0,$$

where the divergence and the norm are computed in $\mathbb{R}^2$, and Gu is a vector field on Ω given in coordinates by $Gu = \nabla u + Z$ where $Z = \tau x_2 \partial_1 - \tau x_1 \partial_2$ and ∇u is the gradient of u in $\mathbb{R}^2$.

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Developing the divergence, one gets the following equation

$$\left(1 + (u_2 - \tau x_1)^2\right) u_{11} - 2(u_1 + \tau x_2)(u_2 - \tau x_1) u_{12} + \left(1 + (u_1 + \tau x_2)^2\right) u_{22} = 0$$

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4. Translationally invariant examples (C. Figueroa, F. Mercuri, R. Pedrosa): $c \in \mathbb{R}$

$$u_c(x_1, x_2) = \tau x_1 x_2 + \frac{\sinh(c)}{4\tau} \left[2\tau x_2 \sqrt{1 + 4\tau^2 x_2^2} + \operatorname{arcsinh}(2\tau x_2) \right]$$

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5. Foliated examples (B. Daniel): $u(x_1, x_2) = x_1 f(x_2)$ where f is a C^2 function on $\mathbb{R}$. The graphs are foliated by Euclidean straight lines (not geodesic in $Nil_3(\tau)$ in general). They are asymptotic to $x_3 = 0$ on one side and to FMP's examples on the other side $(x_2 \longrightarrow \pm\infty)$.

Examples of minimal surfaces in $Nil_3(\tau)$.

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6. Horizontal catenoids (B. Daniel, L. Hauswirth):

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5. $\mathcal{C}_\alpha$ is invariant by rotation of angle π around all the axis.

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Existence on bounded domains

For finite boundary data, on bounded domain the more general existence result is established for convex boundary and piecewise continuous boundary data. Before this result, there were existence result with more restrictive assumptions (L. Alias, M. Dajczer, J.H. De Lira, -, H. Rosenberg, R. Sa Earp, E. Toubiana).

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- These results are in contrast with the $\mathbb{R}^3$ case, where a minimal solutions with zero boundary value on a wedge of angle $< \pi$ is zero (H.Rosenberg, R. Sa Earp).

THEOREM (-, R. SA EARP, E. TOUBIANA)

- Let $\Omega \subset \mathbb{R}^2$ be an unbounded, convex domain different from an half-plane. Let φ be a continuous function on $\Gamma = \partial\Omega$ except at a discrete set of points where φ has left and right limit. Then there exists a minimal extension u of φ over $\bar{\Omega}$. Moreover the boundary of the graph of u contains the vertical segments between the left and the right limits of φ at the discontinuity points.

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- Let Ω be a half-plane and let $\Gamma = \partial\Omega$. Let φ be a bounded function on Γ, continuous except at a discrete set of points where φ has left and right limit. Then there exists a 1-parameter family of minimal extensions *u* of φ over $\bar{\Omega}$. Moreover the boundary of the graph of *u* contains the vertical segments between the left and the right limits of φ at the discontinuity points.

GRAPHS ON UNBOUNDED DOMAINS

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- Let Ω be a half-plane and let $\Gamma = \partial\Omega$. Let φ be a bounded function on Γ, continuous except at a discrete set of points where φ has left and right limit. Then there exists a 1-parameter family of minimal extensions u of φ over $\bar{\Omega}$. Moreover the boundary of the graph of u contains the vertical segments between the left and the right limits of φ at the discontinuity points.

- In the half-plane case, we can relax the assumption on φ. For example, if the half plane is $x_2 > 0$: $\varphi(x_1, 0) = cx_1$ for $|x_1| > n$.

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4. Schauder: $C^{1,\beta}$ implies $C^{2,\alpha}$.
5. Ladyzhenskaya-Ural'ceva: C^1 implies $C^{1,\beta}$.

Main Steps and Tools of the Proof

- C^1 follows from

  Theorem (H. Rosenberg, R. Souam, E. Toubiana) Let $\Omega \subset \mathbb{R}^2$ be a relatively compact domain and let $u : \Omega \longrightarrow \mathbb{R}$ satisfy the minimal surface equation. Then, for any positive constant C_1, C_2, there exists a constant $\alpha = \alpha(C_1, C_2, \Omega)$ such that for any $p \in \Omega$ with $d(p, \partial\Omega) \geq C_2$ and $|u| < C_1$ on Ω, we have

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- **Uniform height estimates implies convergence**
- Then one has to use barrier in order to prove that boundary data are right.

Geometric idea of the proof in the halfplane case

GEOMETRIC IDEA OF THE PROOF IN THE HALFPLANE CASE

- Assume that the half-plane is

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > d\}.$$

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- Exhaust Ω_n by rectangles

$$\mathcal{R}_{n,k} = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < k,\ d < x_2 < n\}$$

and define $\varphi_{n,k}(p) = \varphi_n(p)$ for $p \in \partial\Omega_n \cap \partial\mathcal{R}_{n,k}$, and it is monotone on the vertical sides of $\mathcal{R}_{n,k}$ that is $\{(x_1, x_2) \in \bar{\mathcal{R}}_{n,k},\ \ x_1 = \pm k\}$.

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- One proves that all the solutions u_n and u take the right boundary value at continuity points using the technique of barriers, and at the discontinuity points, by hand.
- The existence of one parameter family of solutions is achieved by changing the slope of the initial plane that one uses as supersolution.

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- The proof uses horizontal catenoids and geometric maximum principle.

Contents

- Nil(τ)
- Examples
- Existence
- **Area Growth**
- Height Growth
- An Open Problem

TWO TYPES OF AREA GROWTH OF A GRAPH G IN $Nil_3(\tau)$

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$$\liminf_{R\to\infty} \frac{\mathrm{Area}(G \cap B(p_0, R))}{R^{\alpha}} > 0, \quad \left(\limsup_{R\to\infty} \frac{\mathrm{Area}(G \cap B(p_0, R))}{R^{\alpha}} < \infty \right)$$

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- Daniel examples: $u(x_1, x_2) = x_1 f(x_2)$ where f is a C^2 function on $\mathbb{R}$. Extrinsic (and cylindrical): cubic. Intrinsic: ???. Conformal type: parabolic.

GEODESIC BALLS IN $Nil_3(\tau)$

LEMMA (M. MANZANO, -)

Given $R > 0$, let $B_R(0)$ be the geodesic ball in $Nil_3(\tau)$ centered at the origin and let $D_R(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < R^2\}$.

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- For R small there were results by C. Jang, J. Park, K. Park (2010).

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$$\mathrm{area}(\Sigma \cap B_R(0)) \leq \int_{\Omega(R)} (1 + |Z|) + h(R)\mathrm{length}(\partial\Omega(R)).$$

where Σ is a graph over Ω by a function u, $\Omega(R) = \Omega \cap D_R(0)$ and $B_R(0) \subset D_R(0) \times [-h(R), h(R)]$ for some positive function h.

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- As a byproduct of the previous inequality: in $\mathbb{R}^3$, the intrinsic area growth is at most quadratic.

Contents

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CONJECTURE

The height growth of an entire minimal graph in $Nil_3(\tau)$ is at most quadratic.

Sketch of the proof of the Height Growth

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- This yields that $1 + |Gu|^2 \leq A^{-1}(1 + r^2)^2$ and this easily gives that $|Gu| \leq B(1 + r^2)$. The height growth easily follows.

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But one knows that the cylindrical area growth is at least as $(R^{\frac{2}{3}})^3$, that gives the desired estimate.

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- If the conjecture is true, then the extrinsic area growth of an entire minimal graph in $Nil_3(\tau)$ would be cubic.

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(Manzano, -)

Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $u \in C^\infty(\Omega)$ be a solution in Ω of the minimal surface equation in $Nil_3(\tau)$, such that $u_{|\partial\Omega} = 0$. Denote $M(r) = \sup_{\rho \leq r} |u|$, then

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- The result is sharp because of the plane and the catenoid (there is a previous non sharp result in $Nil_3(\tau)$ by C. Leandro and H. Rosenberg).

Contents

- Nil(τ)
- Examples
- Existence
- Area Growth
- Height Growth
- **An Open Problem**

An Open Problem (Strong Half-Space Theorem)

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Let Σ be a minimal stable surface in $Nil_3(\tau)$. If the angle function $\nu = \langle E_3, N \rangle$ is such that $\nu^2 \in L^1(\Sigma)$, then Σ is a vertical plane.

- The proof depends on the gradient estimate that we got before that prevents Σ to be a graph when $\nu^2 \in L^1(\Sigma)$. Then we use a classification theorem by J. M. Espinar (2013).

Gracias

Summary

Surface	Curvature	Space	EAG	CAG	IAG	CT
Umbrellas	$H = 0$	$Nil_3(\tau)$	R^3	R^3	R^3	Hyperb.
		$\kappa < 0$	$e^{R\sqrt{-\kappa}}$	$e^{R\sqrt{-\kappa}}$	$e^{R\sqrt{-\kappa}}$	
FMP surfaces	$H = 0$	$Nil_3(\tau)$	R^3	R^3	R^3	Parab.
Ideal Scherk	$4H^2 + \kappa < 0$	$\mathbb{E}(\kappa, \tau)$	$\leq R^2$ $(H = 0)$		$\leq R^2$	
k-noids		$\mathbb{H}^2(\kappa) \times \mathbb{R}$				
Entire graphs	$H = 0$	$Nil_3(\tau)$	$\geq R^2, \leq R^3$	$\geq R^3, \leq R^4$	$\leq R^3$	
	$4H^2 + \kappa = 0$	$\kappa < 0$		$\geq e^{R\sqrt{-\kappa}}$	$\leq R^3$	
	$4H^2 + \kappa < 0$			$\geq e^{R\sqrt{-\kappa}}$	$\leq Re^{R\sqrt{-\kappa-4H^2}}$	
	$H = 0$		$\leq Re^{R\sqrt{-\kappa}}$	$\geq e^{R\sqrt{-\kappa}}$	$\leq Re^{R\sqrt{-\kappa}}$	
Graphs with zero boundary values	$H = 0$	$\mathbb{R}^3$	$\leq R^2$		$\leq R^2$	
		$Nil_3(\tau)$	$\leq R^3$		$\leq R^3$	
		$\kappa < 0$	$\leq Re^{R\sqrt{-\kappa}}$		$\leq Re^{R\sqrt{-\kappa}}$	

TABLE : EAG=Extrinsic Area Growth, CAG=Cylindrical Area Growth, IAG= Intrinsic Area Growth, CT= Conformal Type