Genus-g Helicoids

Brian White (joint work with David Hoffman and Martin Traizet)

June 18, 2013 (Granada)

Goals:

Today: construct many interesting minimal surfaces in $\mathbf{S}^2 \times \mathbf{R}$.

Tomorrow: Let the radius of the $\mathbf{S}^2$ tend to ∞ to get interesting examples in $\mathbf{R}^3$. (Hardest part: control genus).

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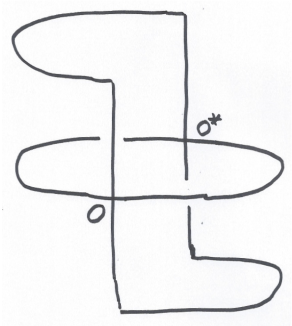
Today's talk

First, I will prove a special, very concrete case of our results about $\mathbf{S}^2 \times \mathbf{R}$.

Then I will indicate how exactly the same proof gives a more general result.

The boundary curve Γ



This curve Γ lies on the boundary $X \times \mathbf{R}$ of a solid cylinder. (Here X is a great circle in $\mathbf{S}^2$.) It consists of the great circle X (at height 0), two vertical segments passing through a pair of diametrically opposite points O and O^* on X, and two horizontal great semicircles at heights h and $-h$.

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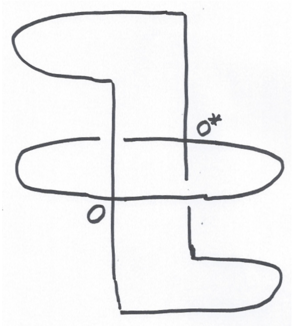

The curve Γ has many symmetries, including reflection μ in a totally geodesic cylinder that switches O and O^*. Throughout this talk, all objects (surfaces, jacobi fields, etc) are required to have all of those symmetries.

To show: Γ bounds many embedded minimal surfaces inside the solid cylinder.

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Main Theorem

Suppose Γ is "bumpy", i.e., bounds no minimal surfaces (inside the cylinder) with nontrivial jacobi fields. Let k be an even integer ≤ 2. Then Γ bounds:

- *an odd number of positive minimal surfaces (inside the cylinder) with Euler characteristic k, and*
- *an odd number of negative minimal surfaces (inside the cylinder) with Euler characteristic k.*

Corollary

Without assuming "bumpiness", the theorem remains true with "an odd number of" replaced by "at least one".

Remark: Repeated Schwarz reflection gives complete, properly embedded minimal surfaces (without boundary) in $\mathbf{S}^2 \times \mathbf{R}$.

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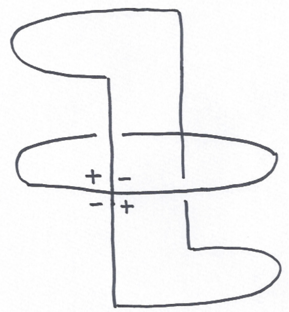
Positive and negative surfaces

In a neighborhood of O, the curve Γ divides the cylinder $X \times \mathbf{R}$ into four quadrants. We call two of the (non-adjacent) quadrants "positive" and the other two "negative":



A surface M (in the solid cylinder) with boundary Γ is called positive it is tangent to the positive quadrants at O, and negative if it is tangent to the negative quadrants at O.

A key tool

Basic Counting Theorem (W 1986)

Let $N \approx \mathbf{B}^3$ be a mean convex, compact Riemannian 3-manifold that contains no closed minimal surfaces. Let C be a smooth, simple closed curve in ∂N. Suppose C is bumpy. Then C bounds

- *an odd number of embedded minimal disks, and*
- *an even number of embedded surfaces of each other topological type.*

(cf. Tomi and Tromba (1978): disks in $\mathbf{R}^3$.)

David and I generalized this theorem to N with piecewise smooth boundary, and to curves and surfaces invariant under a group of symmetries of N.

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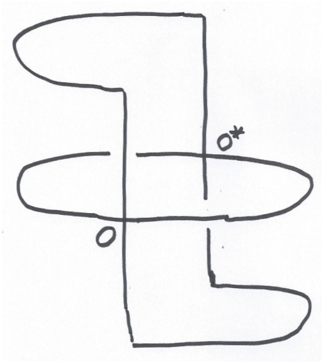
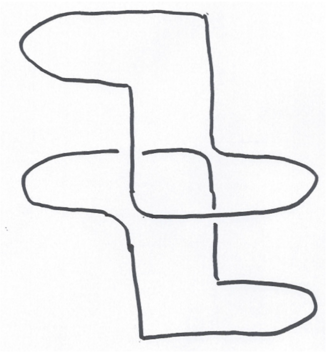
Proof of main theorem

Let $\Gamma(t)$ be obtained from Γ by rounding the corners (as illustrated) to make it embedded. Here $\Gamma(t) \to \Gamma$ as $t \to 0$.



(a) The curve Γ



(b) The rounded curve $\Gamma(t)$

Let $S(t)$ be a one-parameter family of minimal surfaces with $\partial S(t) = \Gamma(t)$.

Then as $t \to 0$, $S(t)$ converges to a surface S bounded by Γ.

Question

How are the topologies (i.e, Euler characteristics) of S and $S(t)$ related?

(Note: here S denotes the open surface: $S = \overline{S} \setminus \Gamma$.)

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It depends on the sign of S:

If S is negative, then S and $S(t)$ are homeomorphic, so $\chi(S) = \chi(S(t))$.

If S is positive, then $\chi(S) = \chi(S(t)) + 2$.

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Let $F(k,+)$ be the number of positive minimal surfaces of Euler characteristic k bounded by Γ. Similarly define $F(k,-)$.

Goal: Show that $F(k,+)$ and $F(k,-)$ are odd for all even $k \leq 2$.

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Claim

For small t,

$$\#(\textit{minimal surfaces bounded by } \Gamma(t) \textit{ with } \chi = k) = F(k,-) + F(k+2,+).$$

For $k < 2$, the first number is even (by the Hoffman-White counting theorem).

Thus

$$F(k,-) \cong F(k+2,+) \mod 2.$$

Similarly

$$F(k,+) \cong F(k+2,-) \mod 2.$$

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The general case

In the preceding argument, the cylinder in $\mathbf{S}^2 \times \mathbf{R}$ can be replaced by any helicoid H in $\mathbf{S}^2 \times \mathbf{R}$.

The argument above shows that the curve Γ bounds positive and negative embedded minimal surfaces S (lying on one side of H) of each Euler characteristic ≤ 2. Reflecting in Z (or equivalently Z^*) gives a surface M bounded by two horizontal great circles, at heights $\pm h$.

Letting $h \to \infty$ gives a properly embedded minimal surface M^* such that

$$M^* \cap H = X \cup Z \cup Z^*.$$

In this way, for each helicoid H, each sign $\pm$, and each even genus g, we get at least one M^* with the specified sign and genus.

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Where are the handles?

When taking limits (as in Martin's talk tomorrow), it is important for us to have the handles lined up along the y-axis, i.e., that the surfaces we produce be "Y-surfaces" (as defined in David's talk yesterday).

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Another key tool: controlling area-blowup

Let M_i be a sequence of minimal surfaces in a Riemannian manifold Ω. Let Z be the area blowup set: Z is the set of points p such that

$$\limsup \operatorname{area}(M_i \cap \mathbf{B}(p,r)) = \infty \qquad \text{for every } r > 0.$$

Theorem (W)

Suppose the boundaries ∂M_i have locally uniformly bounded length:

$$\sup_i \operatorname{length}(U \cap \partial M_i) < \infty \qquad \text{for } U \subset\subset \Omega.$$

Then Z obeys the same maximum principles that hold for properly embedded minimal surfaces without boundary.

In particular, if Z lies on one side of a connected minimal surface M and if Z and M touch at a point, then Z contains M.

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Corollary (Halfspace Theorem)

Suppose $\Omega = \mathbf{R}^3$ and that Z is contained in a half space. Suppose also that Z contains no plane. Then $Z = \emptyset$.

The corollary follows from the theorem because the Hoffman-Meeks proof of their halfspace theorem only uses the maximum principle, and hence the same proof works for area blowup sets Z.

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Example

Here is one example of how we use the area blowup theorem.

When we take a limit M (as sets) of genus g helicoids M_i in $\mathbf{S}^2(R) \times \mathbf{R}$ as $R \to \infty$, we first show that M converges nicely outside of some (possibly very large) solid cylinder C about the z-axis. Thus the area blowup set Z lies in the solid cylinder C.

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Alternate argument

Alternatively, one can argue as follows. Let Σ be a catenoid whose axis is the z-axis and that is disjoint from C. If the area blowup set Z were nonempty, we could shrink Σ until it just touched Z, violating the maximum principle. Hence Z is empty.