An Integral Spectral Representation of the Massive Dirac Propagator in the Kerr Geometry in EF-type Coordinates

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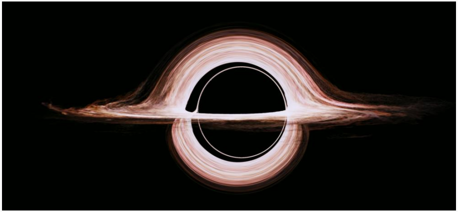

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Geometry Seminar

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Outline of the Talk

Motivation

Study of dynamics of relativistic spin-1/2 fermions in a rotating black hole spacetime via Dirac equation in non-extreme Kerr geometry.

Approaches:

I. Chandrasekhar's mode analysis.
II. Scattering theory.
III. Integral spectral representation of Dirac propagator in Hamiltonian framework.

Investigation of

- decay rates for Dirac spinors.
- probability estimates for Dirac particles to fall into a Kerr black hole or escape to infinity.
- Kerr black hole stability under small fermionic field perturbations.
- scattering and super-radiance.

Problem: Validity of solutions restricted to coordinate domains.

Usual Boyer–Lindquist coordinates singular at horizons.

$\Rightarrow$ Dynamics near and across horizons not well-defined.

Solution: Analytic extension of BLC to advanced Eddington–Finkelstein-type coordinates.

- Regularity at horizons.
- Time function on partial Cauchy surfaces.

Aim 1: Mode analysis in horizon-penetrating coordinates [*C.R., GRG, '17*].

Issues:

I. Trafo singular at horizons. ⇒ Careful mathematical analysis essential!
II. Mixing of variables in trafo leads to symmetry breaking of structures inherent to BLC. ⇒ Separation of variables property conserved?

Aim 2: Generalized horizon-penetrating integral spectral representation of Dirac propagator [*F. Finster & C.R., ATMP, '18*].

Issues:

I. Self-adjointness of Dirac Hamiltonian. ⇒ Dirac Hamiltonian not uniformly elliptic!
II. Construction of well-defined Cauchy problem. ⇒ MIT boundary conditions due to singularity.

Results:

- Proper understanding of black hole evolution and stability.
- Propagation of fermions on black hole backgrounds across horizons.

Preliminaries

The Einstein field equation:

Lorentzian 4-manifold $(\mathfrak{M}, \boldsymbol{g})$ with metric $\boldsymbol{g}\colon T_p\mathfrak{M} \times T_p\mathfrak{M} \to \mathbb{R}$, $p \in \mathfrak{M}$, of signature $(1,3)$ being non-singular, symmetric, bilinear tensor field of type $(0,2)$.

Riemann curvature $\mathrm{Riem}\colon \Gamma^\infty(\mathfrak{M}, T\mathfrak{M})^3 \to \Gamma^\infty(\mathfrak{M}, T\mathfrak{M})$

$$\mathrm{Riem}(\boldsymbol{v}_1, \boldsymbol{v}_2)\boldsymbol{v}_3 := \boldsymbol{\nabla}_{\boldsymbol{v}_1}\boldsymbol{\nabla}_{\boldsymbol{v}_2}\boldsymbol{v}_3 - \boldsymbol{\nabla}_{\boldsymbol{v}_2}\boldsymbol{\nabla}_{\boldsymbol{v}_1}\boldsymbol{v}_3 - \boldsymbol{\nabla}_{[\boldsymbol{v}_1, \boldsymbol{v}_2]}\boldsymbol{v}_3$$

with Levi-Civita connection $\boldsymbol{\nabla}\colon \Gamma^\infty(\mathfrak{M}, T\mathfrak{M})^2 \to \Gamma^\infty(\mathfrak{M}, T\mathfrak{M})$ and $\boldsymbol{v}_i \in T_p\mathfrak{M}$.

Einstein field equations

$$\boldsymbol{G}(\boldsymbol{g}) + \Lambda\, \boldsymbol{g} = 8\pi\, \boldsymbol{T}\,,$$

where

- $\boldsymbol{G}(\boldsymbol{g}) := \mathrm{Ric}(\boldsymbol{g}) - \frac{1}{2}\,\boldsymbol{g}\,\mathrm{Sc}(\boldsymbol{g})$ is Einstein tensor
- Ricci curvature Ric is partial trace of Riem
- scalar curvature $\mathrm{Sc} = \mathrm{tr}_{\boldsymbol{g}}\,\mathrm{Ric}$ is full trace of Riem
- $\boldsymbol{T}$ is energy-momentum tensor of matter and fields
- Λ is cosmological constant.

Einstein field equations

- are nonlinear, inhomogeneous second-order PDE system for $\boldsymbol{g}$.
- are dynamical equations for gravitational field in GR.
- describe gravitation as interaction of geometry with mass-energy content of spacetime.
- imply local conservation of energy and momentum if cosmological constant is vanishing

$$\operatorname{div} \boldsymbol{T} = 0\,.$$

Special case under consideration:

Vanishing energy-momentum tensor $\boldsymbol{T} = \boldsymbol{0}$ and zero cosmological constant $\Lambda = 0$.

$\Rightarrow$ Vacuum Einstein field equations

$$\mathrm{Ric}(\boldsymbol{g}) = \boldsymbol{0}\,.$$

The non-extreme Kerr geometry:

2-parameter family of vacuum solutions of EFE derived under the assumptions of stationarity and axial symmetry.

Physical branch with $\{(M, J) \,|\, 0 \leq J/M < M\}$.

Connected, orientable and time-orientable, smooth, asymptotically flat Lorentzian 4-manifold $(\mathfrak{M}, \boldsymbol{g})$ with topology $S^2 \times \mathbb{R}^2$.

Coordinates $(t, \boldsymbol{x})$ with $t \in \mathbb{R}$ and $\boldsymbol{x} = (x^1, x^2, x^3)$ being coordinates on space-like hypersurface $\mathfrak{N} \simeq S^2 \times \mathbb{R}$.

Metric representation

$$\boldsymbol{g} = a\big(x^1, x^2\big)\, \mathrm{d}t \otimes \mathrm{d}t + b\big(x^1, x^2\big)\big(\mathrm{d}t \otimes \mathrm{d}x^3 + \mathrm{d}x^3 \otimes \mathrm{d}t\big) - \big(\boldsymbol{g}_{\mathfrak{N}}(\boldsymbol{x})\big)_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$$

with

- $a, b \in C^\infty(\mathfrak{M}, T\mathfrak{M})$
- induced Riemannian metric $\boldsymbol{g}_{\mathfrak{N}}$ on $\mathfrak{N}$
- $x^3 = \varphi$ azimuthal angle about axis of symmetry
- Killing vector fields $\boldsymbol{K}_1 = \partial_t$ and $\boldsymbol{K}_2 = \partial_\varphi$.

Application:

Final equilibrium state of gravitational field of uncharged, rotating black holes.

Characteristics and features:

I. According to Carter–Robinson theorem, Kerr solution is unique.

II. Kerr geometry is of Petrov type D.

$\Leftrightarrow$ Weyl tensor has two double eigenbivectors.

$\Leftrightarrow$ Two double principal null directions.

III. Special surfaces: Event horizon, Cauchy horizon, and ergosurface (static limit surface).

IV. Singularity structure: Ring-shaped curvature singularity.

The massive Dirac equation:

Spinor bundle $S\mathfrak{M}$ on globally hyperbolic Lorentzian 4-manifold $(\mathfrak{M}, \boldsymbol{g})$ with fibers $S_p\mathfrak{M} \simeq \mathbb{C}^4$, $p \in \mathfrak{M}$.

Fibers endowed with indefinite inner product of signature $(2,2)$

$$\prec\psi|\phi\succ_p : \; S_p\mathfrak{M} \times S_p\mathfrak{M} \to \mathbb{C}\,, \quad (\psi, \phi) \mapsto \psi^\star \phi \quad \text{for} \quad \psi, \phi \in \mathbb{C}^4\,.$$

Dirac operator

$$\mathcal{D} := \mathrm{i}\,\gamma^\mu \nabla_\mu + \mathcal{B}$$

with general relativistic Dirac matrices (γ^μ), external potential $\mathcal{B}$, and metric connection $\boldsymbol{\nabla}$ on $S\mathfrak{M}$

$$\nabla_\mu = \partial_\mu + \tfrac{1}{8}\,\omega_{\mu\alpha\beta}\big[\gamma^\alpha, \gamma^\beta\big]\,,$$

where $\boldsymbol{\omega}$ is spin connection.

Dirac matrices related to metric $\boldsymbol{g}$ by anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}\,\mathbb{1}_{S_p\mathfrak{M}}\,.$$

General relativistic, massive Dirac equation

$$(\mathcal{D} - m)\psi = \mathbf{0}$$

with Dirac 4-spinor $\psi \in C^{\infty}_{\mathsf{sc}}(\mathfrak{M}, \mathbb{C}^4)$ and invariant fermion rest mass m.

Dirac equation describes dynamics of relativistic, massive spin-1/2 fermions.

Hamiltonian formulation of Dirac equation

$$\mathrm{i}\partial_t \psi = H\psi$$

with Dirac Hamiltonian

$$H := -(\gamma^t)^{-1}(\mathrm{i}\gamma^j \nabla_j + \mathcal{B} - m) - \frac{\mathrm{i}}{8}\,\omega_{t\alpha\beta}\left[\gamma^\alpha, \gamma^\beta\right].$$

The Newman–Penrose formalism and Carter tetrad:

Tetrad formalism with local null basis $(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \overline{\boldsymbol{m}})$, where $\boldsymbol{l}, \boldsymbol{n}$ real-valued and $\boldsymbol{m}, \overline{\boldsymbol{m}}$ a conjugate-complex pair. These satisfy:

- $\boldsymbol{l} \cdot \boldsymbol{l} = \boldsymbol{n} \cdot \boldsymbol{n} = \boldsymbol{m} \cdot \boldsymbol{m} = \overline{\boldsymbol{m}} \cdot \overline{\boldsymbol{m}} = 0$ (null conditions)
- $\boldsymbol{l} \cdot \boldsymbol{m} = \boldsymbol{l} \cdot \overline{\boldsymbol{m}} = \boldsymbol{n} \cdot \boldsymbol{m} = \boldsymbol{n} \cdot \overline{\boldsymbol{m}} = 0$ (orthogonality conditions)
- $\boldsymbol{l} \cdot \boldsymbol{n} = -\boldsymbol{m} \cdot \overline{\boldsymbol{m}} = 1$ (cross-normalization conditions).

Tetrad arranged to reflect symmetries or adapted to certain aspects of underlying spacetime.

$\Rightarrow$ Reduction in number of conditional equations and simplified expressions for primary geometric quantities.

Local, non-degenerate and constant metric

$$\boldsymbol{\eta} = \boldsymbol{l} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{l} - \boldsymbol{m} \otimes \overline{\boldsymbol{m}} - \overline{\boldsymbol{m}} \otimes \boldsymbol{m} \,.$$

Torsion-free Maurer–Cartan equation of structure $\mathrm{d}\boldsymbol{e} = -\boldsymbol{\omega} \wedge \boldsymbol{e}$ in NP formalism:

$$\begin{aligned}
\mathrm{d}\boldsymbol{l} &= 2\,\mathrm{Re}(\epsilon)\,\boldsymbol{n} \wedge \boldsymbol{l} - 2\,\boldsymbol{n} \wedge \mathrm{Re}(\kappa\,\overline{\boldsymbol{m}}) - 2\,\boldsymbol{l} \wedge \mathrm{Re}\big([\tau - \overline{\alpha} - \beta]\,\overline{\boldsymbol{m}}\big) \\
&\quad + 2\mathrm{i}\,\mathrm{Im}(\varrho)\,\boldsymbol{m} \wedge \overline{\boldsymbol{m}} \\
\mathrm{d}\boldsymbol{n} &= 2\,\mathrm{Re}(\gamma)\,\boldsymbol{n} \wedge \boldsymbol{l} - 2\,\boldsymbol{n} \wedge \mathrm{Re}\big([\overline{\alpha} + \beta - \overline{\pi}]\,\overline{\boldsymbol{m}}\big) + 2\,\boldsymbol{l} \wedge \mathrm{Re}(\overline{\nu}\,\overline{\boldsymbol{m}}) \\
&\quad + 2\mathrm{i}\,\mathrm{Im}(\mu)\,\boldsymbol{m} \wedge \overline{\boldsymbol{m}} \\
\mathrm{d}\boldsymbol{m} &= \overline{\mathrm{d}\overline{\boldsymbol{m}}} = (\overline{\pi} + \tau)\,\boldsymbol{n} \wedge \boldsymbol{l} + \big(2\mathrm{i}\,\mathrm{Im}(\epsilon) - \varrho\big)\,\boldsymbol{n} \wedge \boldsymbol{m} - \sigma\,\boldsymbol{n} \wedge \overline{\boldsymbol{m}} \\
&\qquad + \big(\overline{\mu} + 2\mathrm{i}\,\mathrm{Im}(\gamma)\big)\,\boldsymbol{l} \wedge \boldsymbol{m} + \overline{\lambda}\,\boldsymbol{l} \wedge \overline{\boldsymbol{m}} - (\overline{\alpha} - \beta)\,\boldsymbol{m} \wedge \overline{\boldsymbol{m}}\,.
\end{aligned}$$

Spin coefficients constitute connection representation in NP formalism.

General relativistic Dirac equation in NP formalism:

$$\begin{aligned}
(l^\mu \partial_\mu + \varepsilon - \varrho)\mathcal{F}_1 + (\overline{m}^\mu \partial_\mu + \pi - \alpha)\mathcal{F}_2 &= \mathrm{i}m\,\mathcal{G}_1/\sqrt{2} \\
(n^\mu \partial_\mu + \mu - \gamma)\mathcal{F}_2 + (m^\mu \partial_\mu + \beta - \tau)\mathcal{F}_1 &= \mathrm{i}m\,\mathcal{G}_2/\sqrt{2} \\
(l^\mu \partial_\mu + \overline{\varepsilon} - \overline{\varrho})\,\mathcal{G}_2 - (m^\mu \partial_\mu + \overline{\pi} - \overline{\alpha})\,\mathcal{G}_1 &= \mathrm{i}m\,\mathcal{F}_2/\sqrt{2} \\
(n^\mu \partial_\mu + \overline{\mu} - \overline{\gamma})\,\mathcal{G}_1 - (\overline{m}^\mu \partial_\mu + \overline{\beta} - \overline{\tau})\,\mathcal{G}_2 &= \mathrm{i}m\,\mathcal{F}_1/\sqrt{2}\,.
\end{aligned}$$

Carter tetrad: Choose tetrad adapted to

(i) Petrov type of Kerr geometry. $\rightarrow$ Tetrad coincides with two principal null directions of Weyl tensor.

(ii) discrete time and angle reversal isometries of Kerr geometry.

Symmetry of frame

$$\boldsymbol{l} \mapsto -\mathrm{sign}(\Delta)\,\boldsymbol{n}\,, \quad \boldsymbol{n} \mapsto -\mathrm{sign}(\Delta)\,\boldsymbol{l}\,, \quad \boldsymbol{m} \mapsto \overline{\boldsymbol{m}}\,, \quad \overline{\boldsymbol{m}} \mapsto \boldsymbol{m}\,.$$

$\Rightarrow$ Only six independent spin coefficients as solution of first Maurer–Cartan equation of structure.

Horizon-penetrating coordinates:

Kerr metric in BLC

$$(t, r, \theta, \varphi) \quad \text{with} \quad t \in \mathbb{R},\, r \in \mathbb{R}_{>0},\, \theta \in (0, \pi)\,,\ \text{and } \varphi \in [0, 2\pi)$$

given by

$$\boldsymbol{g} = \frac{\Delta}{\Sigma} \left(\mathrm{d}t - a \sin^2(\theta)\, \mathrm{d}\varphi\right) \otimes \left(\mathrm{d}t - a \sin^2(\theta)\, \mathrm{d}\varphi\right) - \frac{\sin^2(\theta)}{\Sigma} \left([r^2 + a^2]\, \mathrm{d}\varphi - a\, \mathrm{d}t\right) \otimes \left([r^2 + a^2]\, \mathrm{d}\varphi - a\, \mathrm{d}t\right) - \frac{\Sigma}{\Delta}\, \mathrm{d}r \otimes \mathrm{d}r - \Sigma\, \mathrm{d}\theta \otimes \mathrm{d}\theta\,,$$

where

- mass M and angular momentum aM for which $0 \leq a < M$
- event and Cauchy horizons $r_{\pm} := M \pm \sqrt{M^2 - a^2}$
- horizon function $\Delta = \Delta(r) := (r - r_+)(r - r_-) = r^2 - 2Mr + a^2$
- $\Sigma = \Sigma(r, \theta) := r^2 + a^2 \cos^2(\theta)\,.$

Relations for time/azimuthal angle and radial coordinate along principal null geodesics

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{r^2+a^2}{\Delta} \quad \Leftrightarrow \quad t = \pm\int\frac{r^2+a^2}{\Delta}\,\mathrm{d}r + c_\pm = \pm r_\star + c_\pm$$
$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \pm\frac{a}{\Delta} \quad \Leftrightarrow \quad \varphi = \pm\int\frac{a}{\Delta}\,\mathrm{d}r + c'_\pm = \pm\frac{a}{r_+-r_-}\ln\left|\frac{r-r_+}{r-r_-}\right| + c'_\pm$$

with

$$r_\star := r + \frac{r_+^2+a^2}{r_+-r_-}\ln|r-r_+| - \frac{r_-^2+a^2}{r_+-r_-}\ln|r-r_-|\,.$$

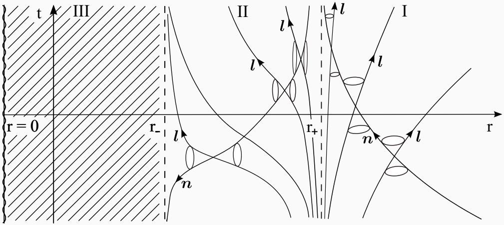
Relations for ingoing principal null geodesics motivate transformation from BLC to advanced Eddington–Finkelstein-type coordinates

$$\mathbb{R}\times\mathbb{R}_{>0}\times(0,\pi)\times[0,2\pi) \to \mathbb{R}\times\mathbb{R}_{>0}\times(0,\pi)\times[0,2\pi)\,, \quad (t,r,\theta,\varphi)\mapsto(\tau,r,\theta,\phi)$$

with

$$\tau := t + r_\star - r = t + \frac{r_+^2+a^2}{r_+-r_-}\ln|r-r_+| - \frac{r_-^2+a^2}{r_+-r_-}\ln|r-r_-|$$
$$\phi := \varphi + \frac{a}{r_+-r_-}\ln\left|\frac{r-r_+}{r-r_-}\right|.$$

AEFTC free of singularities at horizons and spatio-temporal characteristics across horizons conserved.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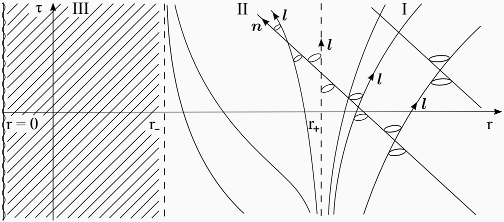
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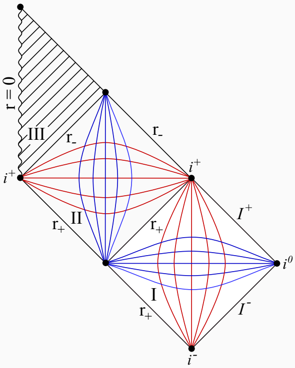

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Kerr metric in AEFTC

$$\begin{aligned}\boldsymbol{g} = &\left(1 - \frac{2Mr}{\Sigma}\right)\mathsf{d}\tau \otimes \mathsf{d}\tau - \frac{2Mr}{\Sigma}\big([\mathsf{d}r - a\sin^2(\theta)\,\mathsf{d}\phi] \otimes \mathsf{d}\tau \\ &+ \mathsf{d}\tau \otimes [\mathsf{d}r - a\sin^2(\theta)\,\mathsf{d}\phi]\big) - \left(1 + \frac{2Mr}{\Sigma}\right)\big(\mathsf{d}r - a\sin^2(\theta)\,\mathsf{d}\phi\big) \\ &\otimes \big(\mathsf{d}r - a\sin^2(\theta)\,\mathsf{d}\phi\big) - \Sigma\,\mathsf{d}\theta \otimes \mathsf{d}\theta - \Sigma\sin^2(\theta)\,\mathsf{d}\phi \otimes \mathsf{d}\phi\,.\end{aligned}$$

Substituting Carter tetrad in AEFTC

$$\begin{aligned}\boldsymbol{l} &= \frac{1}{\sqrt{2\,\Sigma}\,r_+}\big([\Delta + 4Mr]\,\partial_\tau + \Delta\,\partial_r + 2a\,\partial_\phi\big) \\ \boldsymbol{n} &= \frac{r_+}{\sqrt{2\,\Sigma}}(\partial_\tau - \partial_r) \\ \boldsymbol{m} &= \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big) \\ \overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big)\end{aligned}$$

into – and solving – torsion-free Maurer–Cartan equation of structure yields regular spin coefficients.

Chandrasekhar's mode analysis:

Method:

Kerr geometry in NP formalism described by regular Carter tetrad in AEFTC.

Dirac equation in chiral representation in 2-spinor form with NP dyad basis.

Separability into radial and angular ODE systems via weighted mode ansatz.

Analysis of:

I. Asymptotic radial solutions at infinity, event horizon, and Cauchy horizon.
II. Decay of associated errors.
III. Set of eigenfunctions and eigenvalue spectrum of angular system.

Basis for:

- Hamiltonian formulation of massive Dirac equation in non-extreme Kerr geometry in horizon-penetrating coordinates.
- Construction of integral spectral representation of Dirac propagator yielding dynamics outside, across, and inside EH, up to CH.

Separability of Dirac equation via weighted mode ansatz

$$\mathcal{F}_1(\tau, r, \theta, \phi) = \frac{e^{-\mathrm{i}(\omega\tau + k\phi)}}{\sqrt{|\Delta|\,[r - \mathrm{i}a\cos(\theta)]}}\,\mathcal{R}_+(r)\,\mathcal{T}_+(\theta)$$

$$\mathcal{F}_2(\tau, r, \theta, \phi) = \frac{e^{-\mathrm{i}(\omega\tau + k\phi)}}{r_+\sqrt{r - \mathrm{i}a\cos(\theta)}}\,\mathcal{R}_-(r)\,\mathcal{T}_-(\theta)$$

$$\mathcal{G}_1(\tau, r, \theta, \phi) = \frac{e^{-\mathrm{i}(\omega\tau + k\phi)}}{r_+\sqrt{r + \mathrm{i}a\cos(\theta)}}\,\mathcal{R}_-r_+(r)\,\mathcal{T}_+(\theta)$$

$$\mathcal{G}_2(\tau, r, \theta, \phi) = \frac{e^{-\mathrm{i}(\omega\tau + k\phi)}}{\sqrt{|\Delta|\,[r + \mathrm{i}a\cos(\theta)]}}\,\mathcal{R}_+(r)\,\mathcal{T}_-(\theta)\,,$$

where frequency $\omega \in \mathbb{R}$ and wave number $k \in \mathbb{Z} + 1/2$, into first-order radial and angular ODE systems

$$\partial_r \mathcal{R} = \frac{1}{\Delta}\begin{pmatrix} \mathrm{i}\left(\omega\,[\Delta + 4Mr] + 2ak\right) & \sqrt{|\Delta|}\,(\xi + \mathrm{i}mr) \\ \mathrm{sign}(\Delta)\sqrt{|\Delta|}\,(\xi - \mathrm{i}mr) & -\mathrm{i}\omega\,\Delta \end{pmatrix}\mathcal{R}$$

$$\begin{pmatrix} ma\cos(\theta) & -\partial_\theta - \frac{\cot(\theta)}{2} + a\omega\sin(\theta) + k\csc(\theta) \\ \partial_\theta + \frac{\cot(\theta)}{2} + a\omega\sin(\theta) + k\csc(\theta) & -ma\cos(\theta) \end{pmatrix}\mathcal{T} = \xi\mathcal{T}$$

with constant of separation ξ .

Lemma:

Every nontrivial solution of the radial ODE system for $|\omega| > m$ is asymptotically as $r \to \infty$ of the oscillatory form

$$\mathcal{R}(r_\star) = \mathcal{R}_\infty(r_\star) + E_\infty(r_\star) = D_\infty \begin{pmatrix} \mathfrak{f}_\infty^{(1)} \, e^{\mathrm{i}\phi_+(r_\star)} \\ \mathfrak{f}_\infty^{(2)} \, e^{-\mathrm{i}\phi_-(r_\star)} \end{pmatrix} + E_\infty(r_\star) \, ,$$

where $D_\infty, \mathfrak{f}_\infty^{(1/2)}$ are non-zero constants and

$$\phi_\pm(r_\star) := \mathsf{sign}(\omega) \left[-\sqrt{\omega^2 - m^2} \, r_\star + M \left(\pm 2\omega - \frac{m^2}{\sqrt{\omega^2 - m^2}} \right) \ln(r_\star) \right] .$$

The error E_∞ has polynomial decay

$$\| E_\infty(r_\star) \| \leq \frac{a}{r_\star}$$

for a suitable constant $a \in \mathbb{R}_{>0}$. In the case $|\omega| < m$, the non-trivial solution $\mathcal{R}$ has both contributions that show exponential decay $\sim e^{-\sqrt{m^2 - \omega^2} \, r_\star}$ and exponential growth $\sim e^{\sqrt{m^2 - \omega^2} \, r_\star}$.

Lemma:

Every nontrivial solution of the radial ODE system is asymptotically as $r \searrow r_\pm$ of the form

$$\mathcal{R}(r_\star) = \mathcal{R}_{r_\pm}(r_\star) + E_{r_\pm}(r_\star) = \begin{pmatrix} \mathfrak{g}_{r_\pm}^{(1)} \, e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(\pm)}\right) r_\star} \\ \mathfrak{g}_{r_\pm}^{(2)} \end{pmatrix} + E_{r_\pm}(r_\star)$$

with non-zero constants $\mathfrak{g}_{r_\pm}^{(1/2)}$ and an error $E_{r_\pm}$ with exponential decay

$$\|E_{r_\pm}(r_\star)\| \leq p_\pm \, e^{\pm q_\pm r_\star}$$

for r sufficiently close to $r_\pm$ and suitable constants $p_\pm, q_\pm \in \mathbb{R}_{>0}$.

Proposition:

For any $\omega \in \mathbb{R}$ and $k \in \mathbb{Z} + 1/2$, the angular operator has a complete set of orthonormal eigenfunctions $(\mathcal{T}_n)_{n \in \mathbb{Z}}$ in $L^2\big((0,\pi), \sin(\theta)\, \mathrm{d}\theta\big)^2$ that are bounded and smooth away from the poles. The corresponding eigenvalues ξ_n are real-valued and non-degenerate, and can thus be ordered as $\xi_n < \xi_{n+1}$. Both the eigenfunctions and the eigenvalues depend smoothly on ω.

Integral Spectral Representation of the Dirac Propagator

Hamiltonian formulation:

Dirac equation in Hamiltonian form

$$\mathrm{i}\partial_\tau \psi(\tau, \boldsymbol{x}) = H\psi(\tau, \boldsymbol{x}) \quad \text{with} \quad H := -\mathrm{i}(\gamma^\tau)^{-1}\gamma^j \nabla_j + (\text{z.o.t.}) = \alpha^j \, \partial_j + \mathcal{V} \, .$$

Scalar product on space-like hypersurfaces $\mathfrak{N} := \{\tau = \text{const.}, r, \theta, \phi\}$

$$(\psi|\phi)_{\mathfrak{N}} := \int_{\mathfrak{N}} \prec\psi|\boldsymbol{\nu}\!\!\!/\,\phi\succ_p \mathrm{d}\mu_{\mathfrak{N}}$$

with future-directed, time-like normal $\boldsymbol{\nu}$ and invariant measure $\mathrm{d}\mu_{\mathfrak{N}}$ on $\mathfrak{N}$.

Independent of choice of $\mathfrak{N}$. $\Leftarrow$ Gauss' theorem and current conservation.

Requirement for spectral theorem: Self-adjointness of H w.r.t. scalar product.

Evaluation of principal symbol shows H not (uniformly) elliptic at horizons.
$\Rightarrow$ Standard methods of proof from elliptic theory cannot be employed.

New method for mixed initial-boundary value problems combining results from theory of symmetric hyperbolic systems with near-boundary elliptic methods [*F. Finster & C.R., AMSA, '16*].

Self-adjointness of the Dirac Hamiltonian:

Consider subset

$$M := \{\tau, r > r_0, \theta, \phi\} \subset \mathfrak{M} \quad \text{for} \quad r_0 < r_-$$

with time-like boundary

$$\partial M := \{\tau, r = r_0, \theta, \phi\} \,.$$

Foliation of space-like hypersurfaces

$$N = (N_\tau)_{\tau \in \mathbb{R}} \,, \quad \text{where} \quad N_\tau := \{\tau = \text{const.}, r > r_0, \theta, \phi\} \,,$$

with boundaries

$$\partial N_\tau := \partial M \cap N_\tau \simeq S^2 \,.$$

Killing field time-like near ∂M

$$\boldsymbol{K} := \partial_\tau + \beta_0 \, \partial_\phi \quad \text{with constant} \quad \beta_0 = \beta_0(r_0) \in \mathbb{R}\backslash\{0\} \,.$$

Spinor bundle SM of M with fibers $S_pM \simeq \mathbb{C}^4$, where $p \in M$.

Dirac Hamiltonian

$$H = \alpha^j \partial_j + \mathcal{V} \quad \text{on } N \,,$$

symmetric w.r.t. to scalar product $(. | .)_N$.

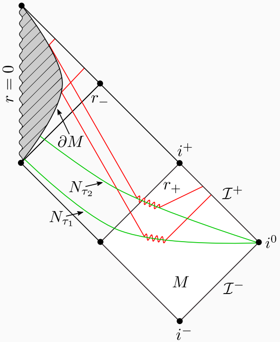
Radial Dirichlet-type boundary condition on time-like hypersurface inside CH

$$(\boldsymbol{\not{n}} - \mathrm{i})\psi_{|\partial M} = \mathbf{0}\,,$$

where $\boldsymbol{n}$ is inner normal to ∂M.

Consequences:

- Dirac particles reflected at ∂M.
- Without effect on dynamics outside CH.
- Unitary time evolution.
- Limiting case where boundary coincides with CH.



Domain of Dirac Hamiltonian

$$\mathrm{Dom}(H) = \left\{\psi \in C_0^\infty(N, SM) \;\middle|\; (\boldsymbol{\not{n}} - \mathrm{i})\psi_{|\partial N} = \mathbf{0}\right\}.$$

Lemma:

The Cauchy problem for the Dirac equation in the Kerr geometry in AEFTC

$$\mathrm{i}\partial_\tau \psi = H\psi\,, \quad \psi_{|\tau=0} =: \psi_0 \in C_0^\infty(N_{\tau=0}, SM)$$

with the radial Dirichlet-type boundary condition at ∂M given by

$$(\not{n} - \mathrm{i})\psi_{|\partial M} = \mathbf{0}\,,$$

where the initial data ψ_0 is smooth, compactly supported outside, across, and inside the event horizon, up to the Cauchy horizon, and is compatible with the boundary condition, i.e.,

$$(\not{n} - \mathrm{i})(H^p \psi_0)_{|\partial N_\tau} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\,,$$

has a unique, global solution ψ in the class of smooth functions with spatially compact support $C^\infty_{\mathrm{sc}}(M, SM)$. Evaluating this solution at subsequent times τ and τ' gives rise to a unique unitary propagator

$$U^{\tau', \tau}\colon C_0^\infty(N_\tau, SM) \to C_0^\infty(N_{\tau'}, SM)\,.$$

Theorem:

The Dirac Hamiltonian H in the non-extreme Kerr geometry in AEFTC with

$$\mathrm{Dom}(H) = \left\{\psi \in C_0^\infty(N_\tau, SM) \;\middle|\; (\not{n} - \mathrm{i})(H^p\psi)_{|\partial N_\tau} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\right\}$$

is essentially self-adjoint.

Spectral theorem, Stone's formula, and derivation of the resolvent:

Derivation of explicit expression for spectral measure in formal spectral decomposition of Dirac propagator

$$\psi(\tau, \boldsymbol{x}) = e^{-\mathrm{i}\tau H}\psi_0(\boldsymbol{x}) = \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau}\psi_0(\boldsymbol{x})\,\mathrm{d}E_\omega$$

with spectral measure $\mathrm{d}E_\omega$ and initial data $\psi_0 \in C_0^\infty\big((r_0,\infty)\times S^2, SM\big)$.

Employing Stone's formula relating spectral projector of H to resolvent yields

$$\psi(\tau, \boldsymbol{x}) = \frac{1}{2\pi\mathrm{i}}\sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k\phi}\int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau}\lim_{\epsilon\searrow 0}\big[\mathsf{Res}^k_{\omega+\mathrm{i}\epsilon} - \mathsf{Res}^k_{\omega-\mathrm{i}\epsilon}\big]\psi_{0,k}(r,\theta)\,\mathrm{d}\omega\,.$$

Computation of resolvents for fixed k-modes on upper/lower complex half-planes:

(i) Factoring out azimuthal angle modes.

(ii) Projecting H onto finite-dim., invariant spectral eigenspace of angular operator from Chandrasekhar's separation of variables.

(iii) Two-dim. radial Green's matrix from Chandrasekhar's separation of variables in terms of Jost-type solutions.

(iv) Summation over angular modes and evaluation of ϵ-limit.

Summary and Outlook

Results: Dirac equation in Kerr geometry in horizon-penetrating coordinates.

I. Chandrasekhar's mode analysis: Separability, radial asymptotics, and spectral properties.
II. Hamiltonian formulation.
III. Essential self-adjointness of Dirac Hamiltonian.
IV. Generalized integral spectral representation of Dirac propagator.

Current research: Formulation of AQFT for Dirac fields in BH spacetimes.

I. Construction of fermionic signature operator.
   - Symmetric operator on solution space of massive Dirac equation in globally hyperbolic spacetimes.
   - Gives rise to (pure, quasi-free) fermionic Fock ground state.
   - Physically sensible provided it is of Hadamard form.
II. Construction of Fock spaces.
   - Construction of ground state via Araki's construction applied to projection operator onto negative spectral subspace of FSO.
   - Anti-commutation relations for creation and annihilation operators.
   - Analysis of resulting many-particle quantum states.

Status quo: Construction of FSO in exterior Schwarzschild geometry and analysis of Fock ground state for Dirac field [*F. Finster & C.R., AHP, '19*].

- Main obstacle: Boundary term for event horizon.
- Finding: Fock ground state coincides with Hadamard state obtained by usual frequency splitting for observer at infinity.

Work in progress:

I. FSO in Kruskal extension of Schwarzschild geometry.
   - Main obstacle: Boundary term for curvature singularity.
   - Expectation: FSO no longer yields frequency splitting. $\Rightarrow$ Thermal Hawking–Unruh state up to spin-gravity coupling corrections.

II. Generalization to non-extreme Kerr geometry.

Overall aim: New derivation of Hawking effect.