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Main Theorem and Proof

Self-adjointness of the Dirac Hamiltonian for a Class of Non-uniformly Elliptic Mixed Initial-boundary Value Problems

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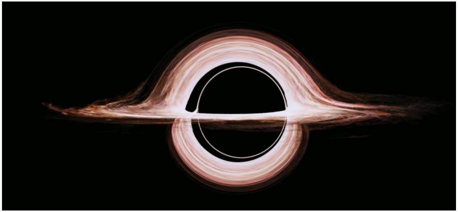

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Geometry Seminar

25th September 2019

Outline of the Talk

I. Motivation

II. Preliminaries

  (i) Geometric setting and assumptions

  (ii) The massive Dirac equation in Hamiltonian form

  (iii) Cauchy problem for the Dirac equation

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Motivation

Study of time-dependent dynamics of relativistic spin-1/2 fermions in analytic extension of non-extreme Kerr geometry across event horizon.

Aims:

- Derivation of Hamiltonian formulation of massive Dirac equation in non-extreme Kerr geometry in horizon-penetrating coordinates [*C.R., GRG, '17; Finster & C.R., ATMP, '18*].
- Construction of integral spectral representation of massive Dirac propagator yielding dynamics outside, across, and inside event horizon, up to Cauchy horizon [*Finster & C.R., ATMP, '18*].

Framework:

Dirac equation in Kerr geometry in Hamiltonian form

$$\mathrm{i}\partial_\tau \psi(\tau, \boldsymbol{x}) = H\psi(\tau, \boldsymbol{x}) \quad \text{with} \quad H := -\mathrm{i}(\gamma^\tau)^{-1}\gamma^j\partial_j + (\text{z.o.t.})\,.$$

Scalar product

$$(\psi|\phi)_{\mathfrak{N}} := \int_{\mathfrak{N}} \prec\psi|\boldsymbol{\psi}\phi\succ_p \, \mathrm{d}\mu_{\mathfrak{N}}\,.$$

Spectral decomposition of Dirac propagator

$$\psi(\tau, \boldsymbol{x}) = e^{-\mathrm{i}\tau H} \psi_0(\boldsymbol{x}) = \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau} \psi_0(\boldsymbol{x}) \, \mathrm{d}E_\omega \, .$$

Requirement:

Self-adjointness of Dirac Hamiltonian for spectral theorem.

Finding:

Dirac Hamiltonian not (uniformly) elliptic at event horizon and Cauchy horizon.
$\rightarrow$ Standard methods of proof from elliptic theory cannot be employed.

Proof of self-adjointness:

New method of proof for general class of non-uniformly elliptic mixed initial-boundary value problems for Dirac equation in smooth and asymptotically flat Lorentzian manifolds, combining results from theory of symmetric hyperbolic systems with near-boundary elliptic methods [*Finster & C.R., AMSA, '16*].

Preliminaries

Geometric setting and assumptions:

Smooth, oriented and time-oriented Lorentzian spin manifold $(\mathscr{M}, \boldsymbol{g})$ of dimension $d \geq 3$ with boundary $\partial\mathscr{M}$.

Assumptions:

(i) $(\mathscr{M}, \boldsymbol{g})$ asymptotically flat with one asymptotic end.

(ii) Existence of Killing field $\boldsymbol{K}$ tangential to – and time-like on – $\partial\mathscr{M}$. It may be space-like or null in $\mathscr{M}\backslash\partial\mathscr{M}$.

(iii) Integral curves γ of $\boldsymbol{K}$, defined by $\dot{\gamma}(t) = \boldsymbol{K}\big(\gamma(t)\big)$, exist for all $t \in \mathbb{R}$.

(iv) Space-like hypersurface $\mathscr{N}$ with compact boundary $\partial\mathscr{N}$ and property that every integral curve γ intersects $\mathscr{N}$ exactly once.

Implications:

- $\mathscr{M}$ and $\partial\mathscr{M}$ have product structures $\mathscr{M} = \mathbb{R} \times \mathscr{N}$ and $\partial\mathscr{M} = \mathbb{R} \times \partial\mathscr{N}$.
- $\boldsymbol{g}$ smooth up to $\partial\mathscr{M}$; inducing $(d-2)$-dim. Riemannian metric on $\partial\mathscr{N}$.

Special cases:

- $(\mathscr{M}, \boldsymbol{g})$ globally hyperbolic if $\partial \mathscr{N} = \emptyset$ and $\mathscr{N}$ complete.
- $(\mathscr{M}, \boldsymbol{g})$ stationary if $\boldsymbol{K}$ time-like in the asymptotic end.

Geometric picture:

Construction of coordinate system $(t, \boldsymbol{x})$, $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathscr{N}$ such that $\boldsymbol{K} = \partial_t$.

$\rightarrow$ Observer co-moving along flow lines of Killing field $\boldsymbol{K}$.

Metric

$$\boldsymbol{g} = a(\boldsymbol{x})\, \mathrm{d}t \otimes \mathrm{d}t + b_i(\boldsymbol{x})\, \mathrm{d}t \otimes \mathrm{d}x^i - \big(\boldsymbol{g}_{\mathscr{N}}(\boldsymbol{x})\big)_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \,,$$

with $a, b_i \in C^\infty(\mathscr{M})$ and $\boldsymbol{g}_{\mathscr{N}}$ induced Riemannian metric on $\mathscr{N}$.

Regions where $\boldsymbol{K}$ is time-like: $a(\boldsymbol{x})$ positive and metric stationary (e.g., near $\partial \mathscr{N}$).

Regions where $\boldsymbol{K}$ is not time-like: $a(\boldsymbol{x})$ may be negative and metric not stationary.

Kerr geometry:

For $r_0 < r_-$ choose

$$\begin{aligned} \mathscr{M} &= \{\tau, r > r_0, \theta, \phi\} \\ \partial\mathscr{M} &= \{\tau, r = r_0, \theta, \phi\} \\ \mathscr{N}_\tau &= \{\tau = \text{const.}, r > r_0, \theta, \phi\}\,. \end{aligned}$$

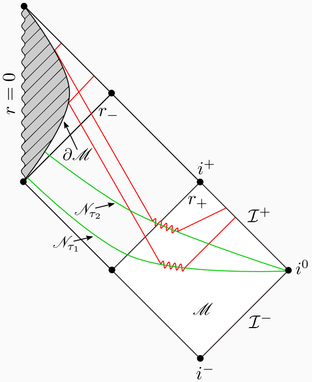
Killing fields $\partial_\tau, \partial_\phi$ not time-like on $\partial\mathscr{M}$.

$\boldsymbol{K} = \partial_\tau + b(r_0)\,\partial_\phi$ is time-like on $\partial\mathscr{M}$ and space-like near spatial infinity.

Dirichlet-type MIT boundary condition on $\partial\mathscr{M}$:

- Reflection condition.
- Shielding of singularity.
- No effect on dynamics outside Cauchy horizon.

$\Rightarrow$ Unitary time evolution.



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The massive Dirac equation in Hamiltonian form:

Spinor bundle $S\mathscr{M}$, i.e., vector bundle with sections $S_p\mathscr{M} \simeq \mathbb{C}^f$, where $p \in \mathscr{M}$ and dimension $f = 2^{\lfloor d/2 \rfloor}$.

Indefinite inner product of signature $(f/2, f/2)$ on $S_p\mathscr{M}$

$$\prec\psi|\phi\succ_p \colon\; S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}\,, \quad (\psi,\phi) \mapsto \psi^\star\phi \quad \text{for} \quad \psi,\phi \in \mathbb{C}^f\,.$$

Dirac operator

$$\mathcal{D} := \mathrm{i}\gamma^\mu \nabla_\mu + \mathcal{B}$$

with

- Dirac matrices (γ^μ); relation to metric via anti-commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}_{S_p\mathscr{M}}$,
- metric connection on spinor bundle $\boldsymbol{\nabla}$,
- external, smooth, matrix-valued potential $\mathcal{B}$; symmetric w.r.t. indefinite inner product.

Dirac equation of mass m

$$(\mathcal{D} - m)\psi = \mathbf{0}\,.$$

Dirac equation in Hamiltonian form

$$\mathrm{i}\partial_t \psi = H\psi$$

with Dirac Hamiltonian

$$H := -\mathrm{i}(\gamma^t)^{-1}\gamma^j \partial_j + (\text{z. o. t.})\,.$$

Taking domain of definition

$$\mathrm{Dom}(H) = C_0^\infty(\mathscr{N} \setminus \partial\mathscr{N}, S\mathscr{M})\,,$$

H symmetric w.r.t. scalar product

$$(\psi|\phi)_{\mathscr{N}} := \int_{\mathscr{N}} \prec\psi|\boldsymbol{\nu\!\!\!/}\phi\succ_p \mathrm{d}\mu_{\mathscr{N}}\,,$$

where $\boldsymbol{\nu}$ is future-directed normal on $\mathscr{N}$ and $\mathrm{d}\mu_{\mathscr{N}}$ volume form on $(\mathscr{N}, \boldsymbol{g}_{\mathscr{N}})$.

Cauchy problem for the Dirac equation:

Existence of unique, global, smooth solution ψ of Cauchy problem

$$\begin{aligned} &\mathrm{i}\partial_t\psi = H\psi \quad \text{in } \mathscr{M} \\ &\psi_{|\mathscr{N}} =: \psi_0 \in \mathrm{Dom}(H) \\ &(\not{n} - \mathrm{i})\psi_{|\partial\mathscr{M}} = \mathbf{0}\,; \quad \boldsymbol{n}\perp\partial\mathscr{M} \end{aligned}$$

with

- domain

$$\mathrm{Dom}(H) = \left\{\psi \in C_0^\infty(\mathscr{N}, S\mathscr{M}) \,\middle|\, (\not{n} - \mathrm{i})(H^p\psi)_{|\partial\mathscr{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\right\},$$

- Dirichlet-type MIT boundary condition with effect that Dirac particles reflected on $\partial\mathscr{M}$.

Essential self-adjointness of the Dirac Hamiltonian:

Finding: H in general not uniformly elliptic.

Ellipticity condition: Principal symbol $P(\boldsymbol{x},\boldsymbol{\xi}) = -\mathrm{i}(\gamma^t)^{-1}\gamma^j\xi_j$ invertible for nonzero $\boldsymbol{\xi}$.

Evaluation of determinant of principal symbol

$$\det\big(P(\boldsymbol{x},\boldsymbol{\xi})\big) = \det\big((\gamma^t)^{-1}\big)\det(\gamma^j\xi_j)\,.$$

Using

$$(\gamma^t)^{-1}(\gamma^t)^{-1} = \frac{\mathbb{1}_{S_p\mathcal{M}}}{g^{tt}} \quad \text{and} \quad \gamma^i\xi_i\,\gamma^j\xi_j = g^{ij}\xi_i\xi_j\,\mathbb{1}_{S_p\mathcal{M}}$$

yields

$$\det\big(P(\boldsymbol{x},\boldsymbol{\xi})\big) = \left(\frac{g^{ij}\xi_i\xi_j}{g^{tt}}\right)^{f/2}.$$

$\Rightarrow$ Hamiltonian fails to be elliptic if $g^{ij}\xi_i\xi_j = 0$ for $\boldsymbol{\xi} \neq \mathbf{0}$.

Consequence: Usual elliptic methods to show self-adjointness of H no longer apply.

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Solution strategy:

I. Splitting solution of Cauchy problem into following contributions:

   a) Region near boundary $\partial\mathscr{M}$ located sufficiently far beyond horizons; standard methods and results for elliptic operators.[1]

   b) Region away from $\partial\mathscr{M}$ including horizons; methods and results from theory of symmetric hyperbolic systems.[2]

II. Adding contributions gives rise to unique, smooth solution of Cauchy problem for small times.

III. Iterating procedure yields global, smooth solution.

IV. Existence of family of unitary time evolution operators.

V. Apply Chernoff's lemma on essential self-adjointness of powers of generators of hyperbolic equations.

[1] R.A. Bartnik and P.T. Chruściel, *Boundary value problems for Dirac-type equations*, arXiv:math/0307278 [math.DG], J. Reine Angew. Math. **579** (2005), 13–73.

[2] M.E. Taylor, *Partial Differential Equations. III*, Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997.

Double Boundary Value Problem for the Dirac Equation

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Preparatory steps for splitting:

Additional boundary condition on suitable surface Y placed near $\partial\mathscr{M}$.

Gaussian normal coordinates in tubular neighborhood of $\partial\mathscr{N}$ in $\mathscr{N}$.

Coordinate system (t, r, Ω), with $t \in \mathbb{R}$, $r \in [0, r_{\max})$, and $\Omega = (\vartheta_1, \ldots, \vartheta_{d-2})$, of $\mathscr{M}$ describing neighborhood of $\partial\mathscr{M}$.

Spacetime region $X := \{(t, r, \Omega) \,|\, 0 \leq r \leq r_{\max}/2\}$ with boundary $\partial X = \partial\mathscr{M} \cup Y$, where $Y := \{(t, r_{\max}/2, \Omega)\}$.

Choice of $r_{\max}$ such that Killing field $\boldsymbol{K}$ time-like in X. $\Rightarrow$ Y time-like surface.

Mixed initial-boundary value problem for Dirac equation

$$\begin{aligned} &\mathrm{i}\partial_t\psi = H\psi \quad \text{in } X \\ &\psi_{|\mathscr{N}} =: \psi_0 \in C^\infty(\mathscr{N} \cap X, S\mathscr{M}) \\ &(\not{n} - \mathrm{i})\psi_{|\partial X} = \mathbf{0} \end{aligned}$$

with $\mathrm{Dom}(H) = \left\{\psi \in W^{1,2}(X \cap \mathscr{N}, S\mathscr{M}) \,\middle|\, (\not{n} - \mathrm{i})\psi_{|\partial X \cap \mathscr{N}} = \mathbf{0}\right\}$.

Proposition:

There is a countable orthonormal basis $(\psi_n)_{n\in\mathbb{N}}$ of eigenfunctions of H with $\psi_n \in \mathsf{Dom}(H)$.

Proof:

Apply abstract spectral theorem given in [*Bartnik & Chruściel, JRAM, '05*].
$\rightarrow$ Task: Verify spectral conditions.

$\square$

Proposition yields spectral decomposition of H.

Proposition implies mixed initial-boundary value problem has unique weak solution in $W^{1,2}(X \cap \mathcal{N}, S\mathscr{M})$ given by

$$\psi(t, \boldsymbol{x}) = \sum_{n=1}^{\infty} c_n \, e^{-\mathrm{i}\omega_n t} \, \psi_n(\boldsymbol{x}) \, , \quad c_n = \int_{X\cap\mathcal{N}} \prec\!\psi_n | \boldsymbol{\psi} \psi_0 \!\succ_p \mathrm{d}\mu_{\mathcal{N}} \, ,$$

where ω_n is eigenvalue of ψ_n.

To apply Chernoff's lemma, one requires solution that is smooth for all times.

Lemma:

Suppose that initial data ψ_0 satisfies the condition

$$(\not{n} - \mathrm{i})(H^p \psi_0)_{|\partial \mathscr{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\,.$$

Then the solution ψ of the mixed initial-boundary value problem is in the class $C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M})$. Conversely, if a solution of the mixed initial-boundary value problem is smooth, then ψ_0 satisfies the above condition.

Solution of the Cauchy Problem

Lemma:

There is an $\varepsilon > 0$ such that the mixed initial-boundary value problem has a unique solution ψ in the class

$$\left\{\psi \in C_0^\infty\big([0,\varepsilon) \times \mathscr{N}, S\mathscr{M}\big) \,\middle|\, (\not{n} - \mathrm{i})(H^p\psi)_{|[0,\varepsilon)\times\partial\mathscr{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\right\}.$$

Proof:

Describe neighborhood of $\partial\mathscr{N}$ via Gaussian normal coordinates.

Decomposition of initial data into contribution ψ_0^{B} near boundary $\partial\mathscr{N}$ and contribution ψ_0^{I} supported in interior of $\mathscr{N}$

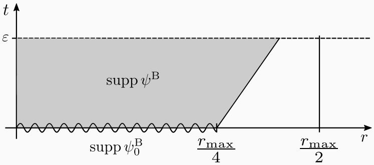
$$\psi_0 = \psi_0^{\mathsf{B}} + \psi_0^{\mathsf{I}}$$

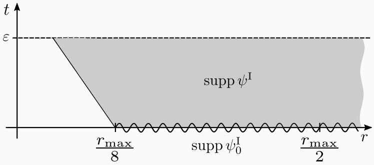
with

- $\psi_0^{\mathsf{B}} := \eta(r)\,\psi_0 \quad$ and $\quad \psi_0^{\mathsf{I}} := \psi_0 - \psi_0^{\mathsf{B}}$,
- test function $\eta \in C_0^\infty\big((-r_{\max}/4, r_{\max}/4)\big) \quad$ and $\quad \eta_{|[0,r_{\max}/8]} \equiv 1$.

Choose ε so small that future development of initial data sets has properties

$$J_{\mathsf{B}}^\vee\big(\{(0,r,\Omega)\,|\,r < r_{\max}/4\}\big) \cap \big(\{\varepsilon\} \times \mathscr{N}\big) \subset \{(\varepsilon,r,\Omega)\,|\,r < r_{\max}/2\}$$
$$J_{\mathsf{I}}^\vee\big(\{(0,r,\Omega)\,|\,r > r_{\max}/8\}\big) \cap \big(\{\varepsilon\} \times \mathscr{N}\big) \subset \{(\varepsilon,r,\Omega)\,|\,r > 0\}.$$





Boundary problem for ψ_0^{B}:

Mixed initial-boundary value problem for Dirac equation

$$\mathrm{i}\partial_t \psi^{\mathsf{B}} = H\psi^{\mathsf{B}} \quad \text{in } X\,, \quad \psi^{\mathsf{B}}_{|\mathcal{N}} = \psi_0^{\mathsf{B}}\,, \quad (\not{n} - \mathrm{i})\psi^{\mathsf{B}}_{|\partial\mathscr{M}\cup Y} = \mathbf{0}\,.$$

Solution $\psi^{\mathsf{B}} \in C^\infty_{\mathsf{sc}}(\mathscr{M}, S\mathscr{M})$ according to previous consideration.

Due to finite propagation speed and specific form of $J_{\mathsf{B}}^{\vee}$, solution vanishes near boundary $\{r = r_{\max}/2\}$, i.e.,

$$\mathrm{supp}\,\psi^{\mathsf{B}}(t,.) \subset [0, r_{\max}/2) \times \partial\mathcal{N} \quad \text{for all} \quad t \in [0,\varepsilon)\,.$$

Extending ψ^{B} by zero leads to global solution in all $\mathscr{M}$.

Interior problem for ψ_0^{I}:

Initial value problem for Dirac equation without boundary conditions

$$\mathrm{i}\partial_t \psi^{\mathsf{I}} = H\psi^{\mathsf{I}} \quad \text{in } \mathscr{M}\backslash\partial\mathscr{M}\,, \quad \psi^{\mathsf{I}}_{|\mathscr{N}} =: \psi_0^{\mathsf{I}}\,.$$

$\mathscr{N}$ complete without boundary and inital data ψ_0^{I} smooth.

$\Rightarrow$ Existence of unique solution $\psi^{\mathsf{I}} \in C^\infty_{\mathsf{sc}}\big([0,\varepsilon)\times\mathscr{N}, S\mathscr{M}\big)$ from fundamental existence and uniqueness theorems of theory of symmetric hyperbolic systems.

Solution vanishes identically near $\partial\mathscr{M}$ due to finite propagation speed as well as specific form of $J_{\mathsf{I}}^{\vee}$.

Full solution:

Adding solutions ψ^{B} and ψ^{I} yields unique, smooth solution ψ of mixed initial-boundary value problem in $C^\infty_0\big([0,\varepsilon)\times\mathscr{N}, S\mathscr{M}\big)$.

Uniqueness of $\psi = \psi^{\mathsf{B}} + \psi^{\mathsf{I}}$ follows from standard energy estimates for symmetric hyperbolic systems.

□

Corollary:

The mixed initial-boundary value problem has unique, global solution ψ in the class

$$\left\{\psi \in C^{\infty}_{\mathsf{sc}}(\mathscr{M}, S\mathscr{M}) \,\middle|\, (\not{n} - \mathrm{i})(H^p \psi)_{|\partial\mathscr{M}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\right\}.$$

The associated time evolution operator

$$U^{t,\,0}\colon C^{\infty}\big(\{0\} \times \mathscr{N}, S\mathscr{M}\big) \to C^{\infty}\big(\{t\} \times \mathscr{N}_t, S\mathscr{M}\big)$$

is unitary with respect to the scalar product $(\,.\,|\,.\,)_{\mathscr{N}}$.

Proof:

Since ε does not depend on initial data, iterate procedure forward and backward in time, obtaining smooth solution for arbitrary positive and negative times.

$\Rightarrow$ Global, smooth solution $\psi \in C^{\infty}_{\mathsf{sc}}(\mathscr{M}, \partial\mathscr{M})$.

Symmetry of H implies scalar product preserved under time evolution.

$\Rightarrow$ Time evolution operator $U^{t,\,0}$ unitary.

$\square$

Main Theorem and Proof

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Main theorem: The Dirac Hamiltonian H with domain of definition

$$\text{Dom}(H) = \left\{\psi \in C_0^\infty(\mathcal{N}, S\mathcal{M}) \,\middle|\, (\not{n} - \mathrm{i})(H^p \psi)_{|\partial\mathcal{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\right\}$$

is essentially self-adjoint.

Proof:

Established results:

- Existence of unique, global, smooth solution of mixed initial-boundary value problem for Dirac equation.
- Existence of unitary time evolution operator $U^{t,\,0}$ defining one-parameter group acting on Dom(H).
- H symmetric with respect to scalar product $(\,.\,|\,.\,)_{\mathcal{N}}$.

Chernoff's lemma[3]: Let T be a symmetric operator with dense domain $\mathrm{Dom}(T) \subset \mathcal{H}$, where $\mathcal{H}$ is a complex Hilbert space. Suppose that T maps $\mathrm{Dom}(T)$ into itself. Suppose in addition that there is a one-parameter group V_t of unitary operators on $\mathcal{H}$ such that $V_t\, \mathrm{Dom}(T) \subset \mathrm{Dom}(T)$, $V_t T = T V_t$ on $\mathrm{Dom}(T)$ and $\partial_t V_t u = \mathrm{i}\, T V_t u$ for $u \in \mathrm{Dom}(T)$. Then every power of T is essentially self-adjoint.

Verify remaining conditions in given framework:

- T corresponds to $-H$ with domain $\mathrm{Dom}(H)$ given in main theorem.
- $\mathrm{Dom}(H)$ invariant under action of H.
- $U^{t,0} H = H U^{t,0}$ is commutativity relation between $e^{-\mathrm{i}tH}$ and H.
- $\partial_t U^{t,0} \psi_0 = -\mathrm{i} H U^{t,0} \psi_0$ is Dirac equation in Hamiltonian form.

$\Rightarrow$ H is essentially self-adjoint on $\mathrm{Dom}(H)$.

$\square$

[3] P.R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Analysis **12** (1973), 401–414.