Study of real hypersurfaces in complex hyperbolic two-plane Grassmannians with Ricci tensors

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Problems

Parallel Ricci tensor

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad (\nabla_X S)Y = 0$$
$$\Longrightarrow M \cong (?)$$

Reeb Parallel Ricci tensor

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad (\nabla_\xi S)Y = 0$$
$$\Longrightarrow M \cong (?)$$

Ricci semi-symmetric

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad R(X,Y) \cdot S = 0$$
$$\Longrightarrow M \cong (?)$$

Contents

Riemannian geometry of $SU_{2,m}/S(U_2 \cdot U_m)$

- $SU_{2,m}/S(U_2 \cdot U_m)$
  : the space of complex two-dimensional linear subspaces though the origin in $\mathbb{C}_2^{m+2}$.
- $SU_{2,m}/S(U_2 \cdot U_m) \approx G/K$
  - $G = SU(2,m)$
    $= \{A \in GL(m+2,\mathbb{C}) \mid \bar{A}^t I_{2,m} A = I, det\ A = 1\}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}.$$

  - $K = S(U(2) \times U(m)) \subset G$
    $= \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in GL(m+2,\mathbb{C}) \mid g_1 \in U(2), \right.$
    $\left. g_2 \in U(m),\ det\ g_1 det\ g_2 = 1 \right\}$
- $dim(SU_{2,m}/S(U_2 \cdot U_m)) = 4m$

- $\mathfrak{g}$ and $\mathfrak{k}$: the Lie algebra of G and K, respectively.
  - $\mathfrak{g} = \mathfrak{su}(m+2)$
    $= \{X \in \mathfrak{gl}(m+2, \mathbb{C}) | \ \bar{X}^t I_{2,m} + I_{2,m} X = 0, \mathrm{tr} X = 0\}$
    $= \mathfrak{k} \bigoplus \mathfrak{m}$
  - Put $o = eK$, where e : the identity of G.
  - $T_o G_2(\mathbb{C}^{m+2}) \approx \mathfrak{m}$
  - $\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{gl}(m+2, \mathbb{C}) \mid A \in \mathfrak{u}(2), \right.$
    $\left. B \in \mathfrak{u}(m), \ \mathrm{tr}(A+B) = 0 \right\}$
    $= \mathfrak{su}(m) \bigoplus \underset{\underset{\mathcal{J}}{\downarrow}}{\mathfrak{su}(2)} \bigoplus \underset{\underset{J}{\downarrow}}{\mathfrak{R}}$, where $\mathfrak{R}$: the center of $\mathfrak{k}$.

$\Longrightarrow SU_{2,m}/S(U_2 \cdot U_m)$ is the unique non-compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathcal{J}$, not containing J.

- $\{J_\nu,\ \nu = 1, 2, 3\}$: a canonical local basis of $\mathcal{J}$ such that
  - $J_\nu^2 = -id,\ \nu = 1, 2, 3$
  - $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where ν is taken modulo 3
  - $\bar{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}$, where q_ν for any $\nu = 1, 2, 3$ are three local one-forms

- (geometric structure)
  J_ν: any almost Hermitian structure in $\mathcal{J}$
  - $JJ_\nu = J_\nu J$
  - JJ_ν: a symmetric endomorphism with $(JJ_\nu)^2 = id$ and $\mathrm{tr}(JJ_\nu) = 0$ where $\nu = 1, 2, 3$

Real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$

Let M be a real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$.

- g : the induced Riemannian metric on M
- ∇ : the Levi-Civita connection of (M, g)
- N : a local unit normal field of M
- A : the shape operator of M w.r.t N, $AX = -\bar{\nabla}_X N$

$J \to (\phi, \xi, \eta, g)$: an almost contact metric structure,
where $JX = \phi X + \eta(X)N$, $\xi = -JN$.

- $\eta(\xi) = 1$, $\phi\xi = 0$, $\eta(\phi X) = 0$, $\phi^2 X = -X + \eta(X)\xi$

$\{J_\nu\}_{\nu=1,2,3} \to (\phi_\nu, \xi_\nu, \eta_\nu, g)$: an almost contact metric 3-structure,
where $J_\nu X = \phi_\nu X + \eta_\nu(X)N$, $\xi_\nu = -J_\nu N$.

- $\eta_\nu(\xi_\nu) = 1$, $\phi_\nu \xi_\nu = 0$, $\eta_\nu(\phi_\nu X) = 0$, $\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu$
- $T_pM = \mathcal{Q} \bigoplus \mathcal{Q}^\perp$, where $\mathcal{Q}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$
- $T_pM = \mathcal{C} \bigoplus \mathcal{C}^\perp$, where $\mathcal{C}^\perp = \text{span}\{\xi\}$

Some fundamental formulas-1

The quaternionic Kähler structure J_ν of $SU_{2,m}/S(U_2 \cdot U_m)$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu \xi_\nu = 0, \\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\
&\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned}$$

for any vector field X tangent to M.

Some fundamental formulas-2

Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$, the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\phi\phi_\nu X = \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu,$$
$$\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu \xi.$$

On the other hand, from the parallelism of Kähler structure J, that is, $\widetilde{\nabla} J = 0$ and the quaternionic Kähler structure $\mathfrak{J}$, together with Gauss and Weingarten formulas it follows that

Some fundamental formulas-3

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$
$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,$$
$$(\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu.$$

- Codazzi equation

$$\begin{aligned}
-2(\nabla_X A)Y + 2(\nabla_Y A)X = \ & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X\} \\
& + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu
\end{aligned}$$

Some fundamental formulas-4

- Gauss equation

$$
\begin{aligned}
-2R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\
& + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
& + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\Big\} \\
& + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\Big\} \\
& - \sum_{\nu=1}^{3}\Big\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\Big\} \\
& - \sum_{\nu=1}^{3}\Big\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\Big\}\xi_\nu \\
& - 2g(AY,Z)AX + 2g(AX,Z)AY,
\end{aligned}
$$

where R denotes the curvature tensor of a real hypersurface M in $SU_{2,m}/S(U_2\cdot U_m)$.

Motivation

There does not exist any Hopf real hypersurface with parallel Ricci tensor in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor, Proc. Royal Soc. Edinb. (2012).

Geometric meaning of parallel

Parallel condition for a (1,1) type tensor field T, $\nabla T = T \otimes \omega$ has a close relation to the holonomy group. The eigenspaces $\{e_i\}_{i=1,\ldots,4m-1}$ are said to be parallel along γ if they are invariant with respect to parallel translation along any curve in M.

S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol.1.

Classification of real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$

Theorem A. Let M be a connected real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{Q}^{\perp}$ are invariant under the shape operator of M if and only if M is locally congruent to an open part of one of the following hypersurfaces:

(A) a tube around a totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$;

(B) a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2 U_{2n})$, $m = 2n$;

(C) a horosphere in $SU_{2,m}/S(U_2 U_m)$ whose center at infinity is singular;

or the following exceptional case holds:

(D) The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by

$$\alpha = \sqrt{2}\ ,\ \gamma = 0\ ,\ \lambda = \frac{1}{\sqrt{2}},$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q})\ ,\ T_\gamma = J(TM \ominus \mathcal{Q})\ ,\ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

J. Berndt and Y. J. Suh, *Hypersurfaces in noncompact complex Grassmannians of rank two*, Internat. J. Math., World Sci. Publ., **23** (2012), 1250103(35 pages).

The typical characterization theorem for $\xi \in \mathcal{Q}$

Theorem B. Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$.
Then the Reeb vector field $\xi \in \mathcal{Q} \Longleftrightarrow M \approx$ *one of Type* (B).

Y.J. Suh, *Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb vector field*, Adv. Appl. Math. **55** (2014), 131–145.

Ricci tensor in differential geometry

- The Ricci curvature tensor S is defined on any pseudo-Riemannian manifold, as a trace of the Riemann curvature tensor.
- S is a symmetric bilinear form on the tangent space of the manifold.
- S provides one way of measuring the degree to which the geometry determined by a given Riemannian metric might differ from that of ordinary Euclidean n-space.
- S represents the amount by which the volume of a geodesic ball in a curved Riemannian manifold deviates from that of the standard ball in Euclidean space.

L. Besse, Einstein manifold, Springer-Verlag (1987).

The Ricci tensor of M in $SU_{2,m}/S(U_2 \cdot U_m)$

The Ricci tensor of M is given by

$$
\begin{aligned}
2SX &= 2\sum_{i=1}^{4m-1} R(X, e_i)e_i \\
&= -(4m+7)X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} \\
&\quad - 2hAX + 2A^2X
\end{aligned}
$$

for any X tangent to M, $h := trace(A)$.

Y.J. Suh and C. Woo, *Real Hypersurfaces in complex hyperbolic two-plane Grassmannians with parallel Ricci tensor*, Math. Nachr. **287** (2014), 1524-1529.

The structure Jacobi operator R_ξ

The structure Jacobi operator R_ξ of M is defined by $R_\xi X = R(X,\xi)\xi$ for any tangent vector $X \in T_pM$, $p \in M$.

$$2R_\xi X = -X + \eta(X)\xi + 2\alpha AX - 2\alpha^2\eta(X)\xi$$
$$+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\},$$

where $\alpha = g(A\xi,\xi)$ is real valued function on M.

Proof

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad R \cdot S = 0$$
$$\Longrightarrow M \cong (?)$$

Lemma 1

Let M be a Ricci semi symmetric Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. Then ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^{\perp}$.

Lemma 2

If A,B,C are diagobalzable matrices and commute with each other, then these exists on a common basis $\{e_k\}_{k=1,\ldots,4m-1}$ which makes A,B,C simultaneously diagonalizable.

Lemma 3

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^{\perp}$, then $SA = AS$.

Lemma 4

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^{\perp}$, then $R_\xi A = AR_\xi$.

Lemma 5

Let M be a Ricci semi-symmetric Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^{\perp}$, then the Ricci tensor S commutes with the structure tensor field ϕ, that is, $S\phi = \phi S$.

Proposition B. [Berndt and Suh, Internat. J. Math.]

Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \perp \mathfrak{J}N$, then one of the following statements holds:

($\mathcal{T}_B$) M has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r),\ \beta = \sqrt{2}\coth(\sqrt{2}r),\ \gamma = 0,$$
$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh(\frac{1}{\sqrt{2}}r),\ \lambda_2 = \frac{1}{\sqrt{2}}\coth(\frac{1}{\sqrt{2}}r),$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = TM \ominus \mathcal{Q},\ T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are invariant under $\mathfrak{J}$ and are mapped onto each other by J. In particular, the quaternionic dimension of $SU_{2,m}/S(U_2U_m)$ must be even.

($\mathcal{H}_B$) M has exactly three distinct constant principal curvatures

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

New Problems

Commuting Ricci tensor 1

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad R_\xi \phi S = S R_\xi \phi$$
$$\Longrightarrow M \cong (?)$$

Commuting Ricci tensor 2

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad \bar{R}_N \phi A = A \bar{R}_N \phi$$
$$\Longrightarrow M \cong (?)$$

GTW parallel Ricci tensor

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad (\widehat{\nabla}^{(k)}_X S) Y = 0$$
$$\Longrightarrow M \cong (?)$$

Proof

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad R_\xi \phi S = S R_\xi \phi$$
$$\Longrightarrow M \cong (?)$$

Lemma 1

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with the commuting condition $R_\xi \phi S X = S R_\xi \phi X$. If the smooth function α is constant along the direction of ξ on M, then the Reeb vector field ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.

Lemma 2

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with $R_\xi \phi S = S R_\xi \phi$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then the Ricci tensor S commutes with the structure tensor field ϕ.

Generalized Tanaka-Webster connection

- Tanaka and Webster(independently) defined Tanaka-Webster (in short, the *GTW*) connection which is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold.
- Tanno introduced the *generalized Tanaka-Webster* connection for contact metric manifolds.
- *Generalized Tanaka-Webster* connection$=$*TW* connection for contact metric manifolds if the associated CR-structure is integrable

GTW connection

Cho defined GTW connection for a real hypersurface of a Kähler manifold by

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + F^{(k)}_X Y,$$

where constant $k \in \mathbb{R} \setminus \{0\}$ and $F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$.

- g : the induced Riemannian metric on M
- ∇ : the Levi Civita connection of (M, g)
- k : a non-zero real number
- A : the shape operator of M w.r.t N

J.T. Cho, CR structures on real hypersurfaces of a complex space form, Publ. Math. Debrecen **54**(1999), 473-487.

J.T. Cho, Levi parallel hypersurfaces in a complex space form, Tsukuba J. Math., **30**(2006), 329-344.

Results

Parallel Ricci tensor

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad (\nabla_X S)Y = 0$$
$$\Longrightarrow \quad \nexists \quad M$$

Reeb Parallel Ricci tensor

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad (\nabla_\xi S)Y = 0$$
$$\Longrightarrow M \cong \quad \mathcal{T}_A \quad or \quad \mathcal{H}_A$$

Ricci semi-symmetric

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad R(X,Y) \cdot S = 0$$
$$\Longrightarrow \quad \nexists \quad M$$

Results: Levi-Civita connection

Commuting Ricci tensor 1

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad R_\xi \phi S = S R_\xi \phi$$
$$\Longrightarrow M \cong \quad \mathcal{T}_A \quad or \quad \mathcal{H}_A$$

Commuting Ricci tensor 2

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad \bar{R}_N \phi A = A \bar{R}_N \phi$$
$$\Longrightarrow M \cong \quad \mathcal{T}_A \quad or \quad \mathcal{H}_A$$

GTW parallel Ricci tensor

$$M \hookrightarrow SU_{2,m}/S(U_2 \cdot U_m) \quad \text{s.t.} \quad (\widehat{\nabla}^{(k)}_X S) Y = 0$$
$$\Longrightarrow \quad \nexists \quad M$$