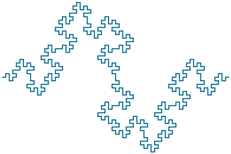
Some new result on sub-Riemannian geodesics in Carnot groups

joint with E. Le Donne, G.P. Leonardi and R. Monti

Davide Vittone

Dipartimento di Matematica - Università di Padova



Granada, October 11th, 2012

Part I

Sub-Riemannian manifolds and geodesics

Sub-Riemannian manifolds

A *sub-Riemannian manifold* is a triple (M, Δ, g) where

- M is a smooth n-dimensional connected manifold;
- $\Delta \subset TM$ is a smooth ("horizontal") sub-bundle of rank m;
- g is a smooth metric on Δ.

A Lipschitz continuous curve $\gamma : [0,1] \to M$ is *horizontal* if

$$\dot{\gamma}(t) \in \Delta_{\gamma(t)} \quad \text{for a.e. } t \in [0,1] .$$

In this case, we can define the *length* $\ell(\gamma) := \int_\gamma |\dot{\gamma}|_g$.

Definition (CC distance)

The *Carnot-Carathéodory distance* between $x, y \in M$ is

$$d_c(x,y) := \inf \{\ell(\gamma) : \gamma : [0,1] \to M \text{ horizontal}, \gamma(0) = x, \gamma(1) = y\} .$$

In general $m < n$ ("sub"-Riemannian).

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Geodesics

If the bracket generating condition

$$\operatorname{rank}\,(\mathfrak{Lie}\,\Delta)(x) = n \quad \forall x \in M$$

holds, then d_c is an actual distance (Chow-Rashevsky).

We are interested in *minimizers*, i.e., horizontal curves $\gamma : [0,1] \to M$ such that

$$\ell(\gamma) = d_c(\gamma(0), \gamma(1)).$$

Minimizers do exist (at least locally). Are they regular?

Without loss of generality we assume that

- $M = \mathbb{R}^n$
- $\Delta = \operatorname{span}\{X_1, \ldots, X_m\}$

where $X_1, \ldots, X_m$ are smooth, linearly independent vector fields in $\mathbb{R}^n$ chosen to be g-orthonormal, i.e.,

$$\text{if} \quad \dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) \quad \text{then} \quad \ell(\gamma) = \int_0^1 |h(t)|dt\,.$$

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Minimizers, extremals and dual curves

Pontryagin maximum principle

Let $\gamma : [0,1] \to \mathbb{R}^n$ be a minimizer such that

$$\gamma(0) = 0 \quad \text{and} \quad \dot{\gamma} = \sum_{j=1}^m h_j X_j(\gamma)\,.$$

Then, there exist

$$\xi_0 \in \mathbb{R}, \qquad \xi : [0,1] \to \mathbb{R}^n \text{ Lipschitz}, \qquad (\xi_0, \xi(t)) \neq (0,0)$$

such that for any $j = 1, \ldots, m$

$$\xi_0 h_j + \langle \xi, X_j(\gamma) \rangle = 0 \quad \text{a.e. on } [0,1]$$

$$\xi_0 h_j + \left\langle \sum_i \xi_i dx^i,\, X_j(\gamma) \right\rangle = 0 \quad \text{a.e. on } [0,1].$$

The function $\xi : [0,1] \to \Lambda^1 \mathbb{R}^n$ is called *dual curve*; it satisfies a certain ODE.

We call *extremal* any curve satisfying the above necessary condition.

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Normal and abnormal geodesics

If $\xi_0 \neq 0$ we say that γ is a *normal* extremal and the equations

$$h_j = -\frac{1}{\xi_0}\langle \xi, X_j(\gamma)\rangle \quad \text{a.e. on } [0,1]$$

yield C^∞-smoothness. Normal extremals are (locally) minimizers. All Riemannian geodesics ($m = n$) are normal; the same holds in *step 2* structures.

If $\xi_0 = 0$, γ is an *abnormal* extremal and satisfies

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Goh condition

Remark. A minimizer/extremal could be both normal and abnormal.

Goh condition

If γ is a strictly abnormal minimizer, then for any $i, j \in \{1, \dots, m\}$

$$\langle \xi, [X_i, X_j](\gamma) \rangle = 0.$$

Equivalently: $\xi(t) \in (\Delta^2_{\gamma(t)})^{\perp}$, where $\Delta^2 := \Delta + [\Delta, \Delta]$.

REGULARITY OF GEODESICS

The regularity of sub-Riemannian geodesics (in fact, of strictly abnormal ones) is one of the main open questions: see the books by Montgomery (2002), Agrachev-Sachkov (2004) and Agrachev-Barilari-Boscain (forthcoming).

It was believed for some time that only normal extremals could be length minimizing (Strichartz 1986, corrected in 1989).

However, strictly abnormal minimizers do exist (Montgomery 1994, Liu-Sussmann 1995, Sussmann 1996) even in Carnot groups (Golé-Karidi 1995). But all these examples are smooth!

On the contrary, abnormal extremals may develop singularities. Leonardi-Monti (2008) prove that extremals with corner-type singularities cannot be minimizers in certain sub-Riemannian structures. Monti (2012) excludes other singularities ("$y = |x|^{3/2}$").

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Part II

Carnot groups

Carnot groups

Definition

A *Carnot group* $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group whose Lie algebra admits the stratification

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

where $V_i = [V_1, V_{i-1}], i = 2, \ldots, s$ (s = "step") and $[V_1, V_s] = \{0\}$.

We endow $\mathbb{G}$ with the sub-Riemannian structure induced by choosing $\Delta := V_1$ and g left-invariant. In particular, we can fix a orthonormal, left-invariant, bracket-generating basis of V_1

$$X_1, \ldots, X_m \, .$$

The induced distance d_c is left-invariant.

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FREE CARNOT GROUPS

A Carnot group is *free* if its Lie algebra has "as few relations as possible". All Carnot groups are quotients of a free Carnot group.

Grayson-Grossmann (1990): a free Carnot group can be identified with $(\mathbb{R}^n, \cdot)$ in such a way that

$$X_j(x) = \mathrm{e}_j + \sum_{i=m+1}^{n} m_{i,j}(x)\, \mathrm{e}_i, \quad j = 1, \dots, m$$

for $m_{i,j}(x)$ suitable monomials.

It can be proved that for any $x \in \mathbb{R}^n$

$$x = (x_1, \dots, x_n) = \exp(x_1 X_1) \circ \exp(x_2 X_2) \circ \cdots \circ \exp(x_n X_n)(0).$$

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An example

A model for the free Carnot group of rank 3 and step 3 is $\mathbb{R}^{14}$ with left invariant vector fields X_1, X_2, X_3

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -x_1 \\ 0 \\ 0 \\ x_1^2/2 \\ x_1x_2 \\ x_1x_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -x_1 \\ -x_2 \\ 0 \\ 0 \\ 0 \\ x_1^2/2 \\ x_1x_2 \\ x_1x_3 \\ x_2^2/2 \\ x_2x_3 \end{bmatrix}$$

Part III

Abnormal geodesics and algebraic varieties in Carnot groups

DUAL CURVE IN CARNOT GROUPS

Let $\mathbb{G}$ be a Carnot group and let θ^j be the base of 1-forms dual to the Hall basis $X_1, \ldots, X_n$:

$$\langle \theta^j, X_i \rangle = \delta_i^j \,.$$

Given an extremal $\gamma(t)$ with dual curve $\xi(t)$, define $\lambda(t)$ by

$$\xi_1 dx^1 + \cdots + \xi_n dx^n = \lambda_1 \theta^1(\gamma) + \cdots + \lambda_n \theta^n(\gamma) \,.$$

For abnormal extremals we have

$$\lambda_1 = \cdots = \lambda_m \equiv 0$$

and for strictly abnormal minimizers the Goh condition reads as

$$\lambda_{m+1} = \cdots = \lambda_{m_2} \equiv 0, \qquad m_2 := \dim \Delta^2 = \dim V_1 + \dim V_2 \,.$$

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Main result

Set $v := \lambda(0) \in \mathbb{R}^n$.

Theorem (Le Donne-Leonardi-Monti-V.)

Let γ be an extremal (either normal or abnormal) in a free Carnot group $\mathbb{G} \equiv \mathbb{R}^n$ with $\gamma(0) = 0$. Then, there exist polynomials $P_1^v, \dots, P_n^v$,

$$P_j^v(x) = \sum_{I \in \mathbb{N}^n} \sum_{\ell=1}^{n} C_{j,I}^{\ell} \, v_\ell \, x^I \qquad (x^I := x_1^{I_1} x_2^{I_2} \cdots x_n^{I_n}),$$

such that $\lambda_j(t) = P_j^v(\gamma(t))$ for any $j = 1, \dots, n$. ▸ Proof

For $j \in \{1, \dots, n\}$ and $I = (I_1, \dots, I_n) \in \mathbb{N}^n$ we have

$$C_{j,I}^{\ell} = \frac{(-1)^{I_1 + \cdots + I_n}}{I_1! \cdots I_n!} \widetilde{C}_{j,I}^{\ell}$$

where the constants $\widetilde{C}_{j,I}^{\ell}$ are defined by

$$[\cdots[[X_j, \underbrace{X_1], \dots, X_1}_{I_1 \text{ times}}], \underbrace{X_2], \dots, X_2}_{I_2 \text{ times}}], \dots] = \sum_{\ell=1}^{n} \widetilde{C}_{j,I}^{\ell} X_\ell \, .$$

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Corollary

A) Let γ be an abnormal extremal in a free Carnot group. Then, there exist polynomials $P_1^v, \dots, P_m^v$ of degree $s-1$ such that

$$P_j^v(\gamma(t)) = 0 \quad \forall j = 1, \dots, m \tag{1}$$

and at least one of them is not the zero polynomial.

B) Let γ be a strictly abnormal minimizer in a free Carnot group. Then, there exist polynomials $P_1^v, \dots, P_m^v, \dots, P_{m_2}^v$ of degree $s-1$ or $s-2$ such that

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and at least two of them are not the zero polynomial.

Remark. The "converse" of A) holds as well. In other words, we provide a characterization of abnormal extremals in free Carnot groups.

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Comments and future directions

Our results could provide a first step towards a “dimensional reduction” argument to attack the problem of minimizers’ regularity. Moreover, they could be useful to classify the possible singularities of abnormal minimizers. One could also try to adapt the tecniques of Leonardi-Monti (2008) to exclude certain singularities for minimizers.

However, we have also some immediate application (to be discussed in a while).

SKETCH OF THE PROOF

Proof. ◂ Statement

Step 1. For any $i,j=1,\dots,n$, the formula

$$X_i P_j^v = \sum_{k=1}^{n} c_{ij}^k P_k^v$$

holds, where $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$.

Step 2. One has

$$P_j^v(\gamma(0)) = v_j = \lambda_j(0) \qquad \text{and} \qquad P_n^v(\gamma(t)) = v_n = \lambda_n(t)\,.$$

Reason by "reverse" induction on j; since $\dot\gamma = \sum_{i=1}^m h_i X_i(\gamma)$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}} P_j^v(\gamma(t)) = \sum_{i=1}^m h_i(t)\, X_i P_j^v(\gamma(t))$$

$$= \sum_{i=1}^m \sum_{\substack{k=1 \\ k \geq j+1}}^n h_i(t) c_{ij}^k P_k^v(\gamma(t)) = \sum_{i=1}^m \sum_{k=1}^n h_i(t) c_{ij}^k \lambda_k(t) = \dot\lambda_j(t).$$

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$$= \sum_{i=1}^{m} \sum_{\substack{k=1 \\ k \geq j+1}}^{n} h_i(t) c_{ij}^k P_k^v(\gamma(t)) = \sum_{i=1}^{m} \sum_{k=1}^{n} h_i(t) c_{ij}^k \lambda_k(t) = \dot\lambda_j(t).$$

Sketch of the proof

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A similar formula

$$\lambda_j(t) = P^v_j(\gamma(t)), \quad P^v_j(x) = \sum_{I\in\mathbb{N}^n} \sum_{\ell=1}^{n} C^\ell_{j,I} v_\ell x^I$$

should hold for extremals in a general Carnot group $\mathbb{G}$. One needs again to work in exponential coordinates of the second type.

Denote by π the omomorphism

$$\pi : \mathbb{G}_{\text{free}} \to \mathbb{G}.$$

If γ is horizontal curve in $\mathbb{G}$, there exists a (essentially) unique horizontal lift κ on $\mathbb{G}_{\text{free}}$.

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Theorem (Tan-Yang, 2011)

Every geodesic γ in a Carnot group $\mathbb{G}$ of step 3 is C^∞-smooth.

Proof. Without loss of generality we may assume that $\mathbb{G}$ is free. Reason by contradiction: then γ is strictly abnormal and contained in a vertical hyperplane

$$\{x \in \mathbb{R}^n : P(x) = a_1x_1 + \cdots + a_mx_m = 0\} \lhd \mathbb{G} \text{ maximal.}$$

Thus

$$\dot\gamma \in \{a_1X_1 + \cdots + a_mX_m\}^\perp \subset V_1,$$

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If γ is an abnormal extremal in a Carnot group of rank $m = 2$ such that

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Then γ is smooth.

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