

Calabi-Bernstein results in Lorentzian products

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Introduction

Introduction

The study of spacelike hypersurfaces in Lorentzian manifolds presents interests in Physics modelling the spatial universe and its matter as well in Mathematics exhibiting new Calabi-Bernstein properties.

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A spacelike hypersurface with zero mean curvature is said to be maximal. These hypersurfaces maximize locally the area functional.

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A. Romero, R. Rubio and J. Salamanca (2014) studied the case where the fiber is parabolic also for the generalized Robertson-Walkers spaces.

Bernsteins Results

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The results and techniques we will present here are in the setting of CMC spacelike hypersurfaces immersed in Lorentzian spaces of the form $-\mathbb{R} \times M$.

Generally speaking they state that, if the growth of the height function on a hypersurface is suitable controlled, then the hypersurface must be a slice.

Lemma (The Omori-Yau generalized maximum principle)

Let Σ^n be an n -dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $f : \Sigma^n \rightarrow \mathbb{R}$ is a smooth function which is also bounded from below. Then there exists a sequence $(p_k) \subset \Sigma^n$ such that

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$$\lim_{k \rightarrow \infty} f(p_k) = \inf f, \quad \lim_{k \rightarrow \infty} |\nabla f(p_k)| = 0 \text{ and } \lim_{k \rightarrow \infty} \Delta f(p_k) \geq 0.$$

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Here such a sequence will be called an Omori-Yau minimizing sequence for the function f , or just Omori-Yau sequence when this term is clear.

The height and support functions

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Geometrically, the support function measures the hyperbolic cosine of the hyperbolic angle between the vectors N and ∂_t , i.e., $\cosh \theta = -\langle N, \partial_t \rangle$.

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where the mean curvature of Σ is $H = -\frac{1}{n}\text{tr}(A)$.

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One equality holds if, and only if, $A = \lambda I$.

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Since X is arbitrary $\nabla f = A(\nabla h)$. The calculus of the Laplacian of the support is more sophisticated and is given in the following lemma.

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where Ric_M is the Ricci curvature of the fiber M^n , $N^* = N + \langle N, \partial_t \rangle \partial_t$ is the projection of N onto M^n and $|A|$ is the Hilbert-Schmidt norm of the shape operator A of Σ^n .

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where \bar{R} is the curvature tensor in the space $-\mathbb{R} \times M^n$ and Σ^n is immersed in that ambience.

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Corollary

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Corollary

Let $\psi : \Sigma \rightarrow -\mathbb{R} \times M^n$ be an immersion with M such that $-\kappa \leq K_M$ and $X \in \mathfrak{X}(\Sigma)$ then

$$Ric(X, X) \geq -\kappa(n-1)(1 + |\nabla h|^2)|X|^2 - \frac{n^2 H^2}{4}|X|^2$$

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Theorem (Albuher, Camargo e de Lima, 2009)

Let $\psi : \Sigma^n \rightarrow -\mathbb{R} \times \mathbb{H}^n$ be an immersed complete spacelike hypersurface with constant mean curvature.

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Then Σ^n must be a slice.

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But firstly we observe another result that arises the same questions.

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$$|\nabla h|^2 \leq \frac{\alpha}{n-1} |A|^2$$

then Σ^n must be an slice, where $0 < \alpha < 1$ and $|A|^2$ is the Hilbert-Schmidt squared norm of the tensor A .

Answer to the first question

Theorem (—, de Lima, 2013)

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$$|\nabla h|^2 \leq \frac{n}{\kappa(n-1)} H^2. \quad (1)$$

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$$|\nabla h|^2 \leq \frac{n}{\kappa(n-1)} H^2. \quad (1)$$

Then, Σ^n is a slice.

Proof (sketch)

By hypothesis we have (1)

$$\langle N, \partial_t \rangle^2 = 1 + |\nabla h|^2 \leq 1 + \frac{n}{\kappa(n-1)} H^2.$$

Consequently, $\inf_{p \in \Sigma} \langle N, \partial_t \rangle(p)$ exists and is negative.

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$$\Delta \langle N, \partial_t \rangle \leq n(n-1)(H^2 - H_2) \langle N, \partial_t \rangle. \quad (2)$$

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Using the Lemma (2) and the Omori-Yau maximum principle we obtain a sequence of points $p_k \in \Sigma^n$ such that

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Using the Lemma (2) and the Omori-Yau maximum principle we obtain a sequence of points $p_k \in \Sigma^n$ such that

$$\lim_k (H^2 - H_2)(p_k) = 0.$$

Since $|A|^2 = nH^2 + n(n-1)(H^2 - H_2)$, we have $\lim_k |A|^2(p_k) = nH^2$.

Proof (sketch)

Reminding that $|A|^2 = \sum_i \kappa_i^2$, for κ_i the eigenvalues of A .
Then, up to subsequence, for $1 \leq i \leq n$ we have $\lim_k \kappa_i(p_k) = \kappa_i^*$ for $\kappa_i^* \in \mathbb{R}$.

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Note that $H = -\frac{1}{n} \sum_i \kappa_i^*$. Then $H^2 = \overline{H}_2$ and for $1 \leq i \leq n$ we have $\kappa_i^* = -H$.

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For $H^2 > 0$ we have $\lim_k \lambda_i(p_k) = 0$. Therefore, $\lim_k |\nabla h|(p_k) = 0$ and this way

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For $H^2 > 0$ we have $\lim_k \lambda_i(p_k) = 0$. Therefore, $\lim_k |\nabla h|(p_k) = 0$ and this way $\inf_{p \in \Sigma} \langle N, \partial_t \rangle(p) = \lim_k \langle N, \partial_t \rangle(p_k) = -1$. ■

Using Bochner-Lichnerowicz's formula

Consider the Bochner-Lichnerowicz's formula

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Using the fact that $|A|^2 = nH^2 + n(n-1)(H^2 - H_2)$ we get

$$\begin{aligned} \frac{1}{2}\Delta \sinh^2 \theta &= nH^2 \cosh^2 \theta + \langle A^2 \nabla h, \nabla h \rangle + \cosh^2 \theta \text{Ric}_M(N^*, N^*) \\ &\quad + n(n-1)(H^2 - H_2) \cosh^2 \theta \end{aligned}$$

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All three parcels in the right hand of the inequality are non-negative. Therefore on a maximizing sequence they converge to zero.

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$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)} |A|^2. \quad (3)$$

Then, Σ^n is a slice $\{t\} \times M^n$.

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Theorem (de Lima,—)

Let $\overline{M}^{n+1} = -\mathbb{R} \times M^n$ be a Lorentzian product space, such that the sectional curvature K_M of its Riemannian fiber M^n satisfies $K_M \geq -\kappa$, for some positive constant κ . Let $\Sigma(u)$ be an entire H -graph over M^n , with u bounded and H_2 bounded from below. If



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$$|Du|_M^2 \leq \frac{|A|^2}{\kappa(n-1) + |A|^2}, \quad (4)$$



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then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.



Example

Consider $u : \mathbb{H}^2 \rightarrow \mathbb{R}$ given by $u(x, y) = a \ln y$, and the graph

$$\Sigma^2(u) = \{(a \ln y, x, y); y > 0\} \subset -\mathbb{R} \times \mathbb{H}^2.$$

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$$|\nabla h|^2 = \frac{|Du|^2}{1 - |Du|^2} = \frac{|a|^2}{1 - |a|^2}.$$

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We also have the mean curvature H of $\Sigma^2(u)$ given by

$$nH = \text{Div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right),$$

Where Div is the divergent in \mathbb{H}^2 .

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$$H = -\frac{a}{2\sqrt{1 - a^2}}$$

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Moreover, we have $H_2 = 0$ on $\Sigma^2(u)$. One can also see that this graph is stable.

In the Half Space

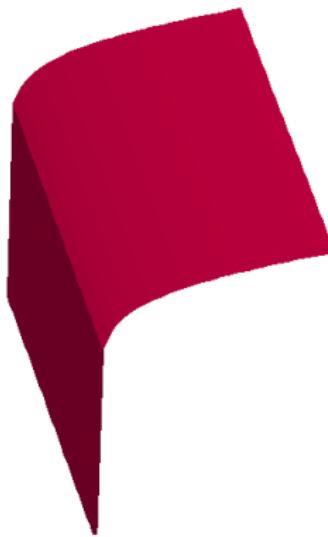


Figura : Stable Graph

In the Poincaré Disc

Plotting it in the Poincaré Disc we obtain something looking like a giraffe. Whose equation is:

$$\ln(-(x^2)/((1-x)^2+y^2)-(y^2)/((1-x)^2+y^2)+(1)/((1-x)^2+y^2))$$

In the Poincaré Disc

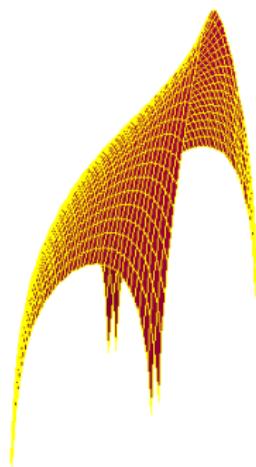


Figura : Plot in the Poincaré Disc

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¡Muchas Gracias!