Asymptotic Dirichlet problems for the mean curvature operator

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Based on joint works with Jean-Baptiste Casteras and Ilkka Holopainen

In $\mathbb{R}^n$ the harmonic (and p-harmonic) functions satisfy the global Harnack's inequality and therefore entire and positive harmonic functions are constant.

Similarly, by a result due to Bombieri, De Giorgi and Miranda, entire and positive solutions of the minimal graph equation are constants. Moreover, we have the Bernstein's theorem

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Every entire solution $u\colon \mathbb{R}^n \to \mathbb{R}$, $n \le 7$, of the minimal graph equation is affine.

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In order to study the existence of entire harmonic functions on Riemannian manifolds, Choi defined the asymptotic Dirichlet problem in 1984. After Choi the existence was studied e.g. by Anderson, Sullivan, Schoen, Cheng, Hsu, March and Borbély.

The study of the non-linear setting of p-harmonic functions started by Pansu in 1988 and was continued by Holopainen and Vähäkangas.

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Cartan-Hadamard manifolds

Definition 2

Cartan-Hadamard manifold is a complete and simply connected Riemannian manifold with non-positive sectional curvature.

Example 3

- $\mathbb{R}^n$, curvature $K = 0$,
- Hyperbolic space $\mathbb{H}^n$, curvature $K = -1$,
- Model manifolds $(\mathbb{R}^n, dr^2 + f(r)^2 d\theta^2)$ with $f'' \geq 0$.

By Cartan-Hadamard theorem, Cartan-Hadamard manifolds are diffeomorphic to $\mathbb{R}^n$ ($\exp_p : T_pM \to M$ is a covering map).

From now on M will denote n-dimensional ($n \geq 2$) Cartan-Hadamard manifold.

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Sphere at infinity

Let $\gamma_1, \gamma_2 \colon \mathbb{R} \to M$ be unit speed geodesics. Then $\gamma_1 \sim \gamma_2$ if

$$\sup_{t \geq 0} d\big(\gamma_1(t), \gamma_2(t)\big) < \infty.$$

The *sphere at infinity* (asymptotic boundary) $\partial_\infty M$ is the set of all equivalence classes of unit speed geodesics.

Equivalently, $\partial_\infty M$ is the set of all unit speed geodesic rays starting from a fixed point $o \in M$, interpretation $\partial_\infty M = \mathbb{S}^{n-1} \subset T_o M$.

Equipping $\bar{M} = M \cup \partial_\infty M$ with the *cone topology*, $\bar{M}$ is homeomorphic to the closed unit ball $\bar{B}^n(0,1) \subset \mathbb{R}^n$ and $\partial_\infty M$ homeomorphic to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

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Cone topology

$\forall x \in M$, $y \in \bar{M} \setminus \{x\}$ there exists a unique unit speed geodesic $\gamma^{x,y} \colon \mathbb{R} \to M$ s.t. $\gamma^{x,y}(0) = x$ and $\gamma^{x,y}(t) = y$ for some $t \in (0, \infty]$. For $x \in M$, $y, z \in \bar{M} \setminus \{x\}$, denote

$$\sphericalangle_x\left(\dot{\gamma}_0^{x,y}, \dot{\gamma}_0^{x,z}\right)$$

the angle between vectors $\dot{\gamma}_0^{x,y}, \dot{\gamma}_0^{x,z} \in T_xM$.
If $v \in T_xM$, define the cone by

$$C(v, \alpha) = \{y \in \bar{M} \setminus \{x\} \colon \sphericalangle_x(v, \dot{\gamma}_0^{x,y}) < \alpha\}$$

and the truncated cone by

$$T(v, \alpha, R) = C(v, \alpha) \setminus \bar{B}(x, R), \quad R > 0.$$

All cones and open balls forms a basis for the cone topology. Details can be found from Eberlein & O'Neill - Visibility manifolds (1973).

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Asymptotic Dirichlet problem

Let $\theta\colon \partial_\infty M \to \mathbb{R}$ be continuous. Then the *asymptotic Dirichlet problem* for minimal graph equation is to find (unique) $u \in C(\bar{M}) \cap C^\infty(M)$ such that

$$\begin{cases} \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 & \text{in } M, \\ u|\partial_\infty M = \theta. \end{cases}$$

In fact, we are looking for the *minimal submanifold* $\Sigma_u \subset M \times \mathbb{R}$ which is the graph of u with "asymptotic boundary" $\{(x, \theta(x)) : x \in \partial_\infty M\} \subset \partial_\infty M \times \mathbb{R}$.

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In the case of $\mathcal{A}$-harmonic functions the equation is

$$-\mathrm{div}\, \mathcal{A}_x(\nabla u) = 0,$$

where $\mathcal{A}\colon TM \to TM$ is an operator that satisfies $\langle \mathcal{A}(V), V \rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2}\mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

In the special case $\mathcal{A}(v) = |v|^{p-2}v$, $\mathcal{A}$-harmonic functions are called p-harmonic and, in particular, if $p = 2$, we obtain the usual harmonic functions.

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How to solve the asymptotic Dirichlet problem, i.e. prove the existence of entire solutions $u\colon \bar{M} \to \mathbb{R}$ with prescribed boundary behaviour $\varphi\colon \partial_\infty M \to \mathbb{R}$?

- Extend the boundary data to a function $\varphi \in C(\bar{M})$.
- Consider an exhaustion of M by geodesic balls $B(o,k)$ and solve the Dirichlet problem with boundary data $u_k|\partial B(o,k) = \varphi|\partial B(o,k)$.
- Extract a converging subsequence and show that the limit has the right boundary values on $\partial_\infty M$.

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Recent results

Some recent results for the minimal graph equation, similar results hold also for harmonic and $\mathcal{A}$-harmonic equations.

Theorem 4 (Casteras-Holopainen-Ripoll (2015))

Let M be a Cartan-Hadamard manifold of dimension $n \geq 3$. Assume that

$$-\frac{\big(\log r(x)\big)^{2\tilde{\varepsilon}}}{r(x)^2} \leq K(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad r(x) \geq R_0,$$

for some $\varepsilon > \tilde{\varepsilon} > 0$. Then the asymptotic Dirichlet problem for the minimal graph equation is uniquely solvable for any boundary data $\theta \in C(\partial_\infty M)$.

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Here the upper bound is (almost) optimal, but also lower bound $\to 0$.

Theorem 5 (Casteras-H-Holopainen (2017))

Let M be a rotationally symmetric n-dimensional Cartan-Hadamard manifold whose radial sectional curvatures outside a compact set satisfy the upper bounds

$$K(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n = 2,$$

and

$$K(P_x) \leq -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n \geq 3.$$

Then the asymptotic Dirichlet problem for the minimal graph equation is solvable with any continuous boundary data on $\partial_\infty M$.

The 2-dimensional case was proved by Ripoll and Telichevesky (2012).

Theorem 6 (Casteras-H-Holopainen (2017))

Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$, let $\phi > 1$ and assume that

$$K(P_x) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \quad r(x) \geq R_0.$$

Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P_x)| \leq C_K |K(P'_x)|$$

whenever $x \in M \setminus B(o, R_0)$. Moreover, suppose that the dimension n and the constant ϕ satisfy the relation

$$n > \frac{4}{\phi} + 1.$$

Then the asymptotic Dirichlet problem for the minimal graph equation is uniquely solvable for any boundary data $\theta \in C(\partial_\infty M)$.

Idea of the proof

Remember that $\partial_\infty M$ is homeomorphic to the unit sphere. Hence, given $\theta \in C(\partial_\infty M)$, we may interpret it as $\theta \in C(\mathbb{S}^{n-1})$.

Suppose first that θ is L-Lipschitz and extend it radially to $\theta \in C(M \setminus \{o\})$. Then $\theta \sim$ "angular function", and the curvature bounds imply that

$$|\nabla\theta(x)| \leq \frac{L}{j(r(x))} \leq \frac{L}{cr(x)^\phi},$$

where $\phi > 1$ and $j(r(x))$ is the infimum of $|V(r(x))|$ over Jacobi fields V, along geodesic $\gamma^{o,x}$, with $V_0 = 0$, $|V_0'| = 1$, $V_0' \perp \dot\gamma^{o,x}$.

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Sketch of the proof

Take a sequence $B_i = B(o, r_i)$, $r_i \nearrow \infty$, and solve the Dirichlet problem

$$\begin{cases} \operatorname{div} \frac{\nabla u_i}{\sqrt{1+|\nabla u_i|^2}} = 0 \quad \text{in } B_i \\ u_i | \partial B_i = \theta. \end{cases}$$

Applying interior gradient estimate and regularity theory of elliptic PDEs, we get a converging subsequence

$$u_{i_k} \to u \quad \text{in } C^2_{loc}(M).$$

It follows that the limit u is a smooth solution to the minimal graph equation on M, and we are left to show (the hard part) that

$$\lim_{x \to x_0} u(x) = \theta(x_0) \quad \forall x_0 \in \partial_\infty M. \tag{1}$$

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Sketch of the proof

To prove (1), we take an auxiliary smooth homeomorphism $\varphi\colon [0,\infty) \to [0,\infty)$ that satisfies certain conditions, denote

$$h = \frac{|u-\theta|}{\nu}, \quad \nu \text{ sufficiently large constant,}$$

and show that

$$\varphi\big(h(x)\big) \to 0 \quad \text{as } x \to x_0 \in \partial_\infty M. \tag{2}$$

We prove (2) by showing

$$\int_M \varphi(h) < \infty$$

and

$$\sup_{B(x,r)} \varphi(h)^{n+1} \leq c \int_{B(x,2r)} \varphi(h).$$

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$$\left.\begin{array}{l}\text{Caccioppoli-type inequality}\\ \text{Weighted Poincaré inequality}\end{array}\right\} \implies \int_M \varphi(h) < \infty$$

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$$\left.\begin{array}{l}\text{Caccioppoli-type inequality}\\ \text{Sobolev inequality}\end{array}\right\} \xRightarrow{\text{Moser iteration}} \sup_{B(x,r)} \varphi(h)^{n+1} \le c \int_{B(x,2r)} \varphi(h).$$

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Caccioppoli-type inequality

Lemma 7

Suppose $\varphi\colon [0,\infty) \to [0,\infty)$ is a homeomorphism that is smooth on $(0,\infty)$ and let $U \subset\subset M$ be open. Suppose that $\eta \geq 0$ is a $C^1(U)$ function and let $u, \theta \in L^\infty(U) \cap W^{1,2}(U)$ be continuous functions such that $u \in C^2(U)$ is a solution to the minimal graph equation in U. Denote $h = |u - \theta|/\nu$, where $\nu > 0$ is a constant, and assume that

$$\eta^2 \varphi(h) \in W_0^{1,2}(U).$$

Then we have

$$\int_U \eta^2 \varphi'(h) \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \leq C_\varepsilon \int_U \eta^2 \varphi'(h) |\nabla \theta|^2 + (4+\varepsilon)\nu^2 \int_U \frac{\varphi^2}{\varphi'}(h) |\nabla \eta|^2$$

for any fixed $\varepsilon > 0$.

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$$\eta^2 \varphi(h) \in W_0^{1,2}(U).$$

Then we have

$$\int_U \eta^2 \varphi'(h) \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \leq C_\varepsilon \int_U \eta^2 \varphi'(h) |\nabla\theta|^2 + (4+\varepsilon)\nu^2 \int_U \frac{\varphi^2}{\varphi'}(h) |\nabla\eta|^2$$

for any fixed $\varepsilon > 0$.

Caccioppoli-type inequality

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Proof.

Consider an auxiliary function

$$f = \eta^2 \varphi\left(\frac{(u-\theta)^+}{\nu}\right) - \eta^2 \varphi\left(\frac{(u-\theta)^-}{\nu}\right),$$

and use it as a test function in

$$\int_U \frac{\langle \nabla u, \nabla f\rangle}{\sqrt{1+|\nabla u|^2}} = 0.$$

$\square$

Weighted Poincaré inequality

The curvature upper bound implies

$$\left.\begin{array}{c} K \le 0, \ \text{everywhere} \\ K(P_x) \le -\frac{\phi(\phi-1)}{r(x)^2}, \, r(x) \ge R_0 \end{array}\right\} \Rightarrow r\Delta r \ge \begin{cases} n-1, & \text{everywhere} \\ \frac{(n-1)\phi}{1+\varepsilon} =: C_0, & r(x) \ge R_0. \end{cases}$$

Using this and Green's formula, we obtain

$$(1+C_0)\int_B \varphi(h) \le c + \int_B r\varphi'(h)|\nabla h|.$$

Then using the Caccioppoli-type inequality twice and applying Young's inequality, we get

$$(C_0 - 4 - \varepsilon')\int_B \varphi(h) \le c + c\int_B F(r\nabla\theta|) + c\int_B F_1(r^2|\nabla\theta|^2) \le C < \infty,$$

where F, F_1 are Young functions and C independent of B.

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f-minimal graphs

Let M be a complete non-compact n-dimensional Riemannian manifold with the Riemannian metric given by $ds^2 = \sigma_{ij}dx^i dx^j$ in local coordinates.

Equip $N = M \times \mathbb{R}$ with the product metric $ds^2 + dt^2$ and let $f \colon N \to \mathbb{R}$ be a smooth function.

A hypersurface $\Sigma \subset M \times \mathbb{R}$ is called f-minimal if its mean curvature satisfies

$$H = \langle \bar{\nabla} f, \nu \rangle$$

at every point of Σ. Here ν denotes the downward pointing unit normal of the surface and $\bar{\nabla} f$ is the gradient with respect to the product metric.

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Examples of f-minimal hypersurfaces:

- minimal hypersurfaces if f is identically constant
- self-shrinkers in $\mathbb{R}^{n+1}$ if $f(x) = |x|^2/4$
- minimal hypersurfaces of weighted manifolds $M_f = (M, g, e^{-f} d\mathrm{vol}_M)$, where (M, g) is a complete Riemannian manifold with volume element $d\mathrm{vol}_M$.

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The Dirichlet problem for f-minimal graphs is to find a solution $u\colon M \to \mathbb{R}$ to the equation

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \langle \bar{\nabla} f, \nu \rangle \quad \text{in } \Omega \\ u|\partial\Omega = \varphi, \end{cases} \tag{3}$$

where $\Omega \subset M$ is a bounded domain and ν denotes the downward pointing unit normal to the graph of u, i.e.

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The equation (3) can be written in non-divergence form as

$$\frac{1}{W}\left(\sigma^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} = \langle \bar{\nabla} f, \nu \rangle,$$

where $W = \sqrt{1 + |\nabla u|^2}$, (σ^{ij}) is the inverse matrix of (σ_{ij}),

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Theorem 8 (Casteras-H-Holopainen (2016))

Let $\Omega \subset M$ be a bounded domain with $C^{2,\alpha}$ boundary $\partial\Omega$. Assume that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ satisfies $f(x,t) = m(x) + r(t)$ with

$$F = \sup_{\bar{\Omega} \times \mathbb{R}} |\bar{\nabla} f| < \infty, \quad Ric_\Omega \geq -\frac{F^2}{n-1}, \quad H_{\partial\Omega} \geq F,$$

where Ric_Ω is the Ricci curvature of Ω and $H_{\partial\Omega}$ is the inward mean curvature of $\partial\Omega$.

Then, for all $\varphi \in C^{2,\alpha}(\partial\Omega)$, there exists a solution $u \in C^{2,\alpha}(\bar{\Omega})$ to the Dirichlet problem (3) *with boundary values φ.*

Remark

If the function f depends on the $\mathbb{R}$-variable we don't have uniqueness for the solutions since the comparison principles fail to hold!

Example 9

Let $B(0,2) \subset \mathbb{R}^2$ be the open disk and $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,t) = \frac{|(x,t)|^2}{4}.$$

Then the upper and lower hemispheres and the disk $B(0,2)$ itself are f-minimal surfaces with zero boundary values on $\partial B(0,2)$.

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The proof is based on the Leray-Schauder method which reduces the solvability to three steps:

I Estimation of $\sup_{\Omega} |u|$

II Estimation of $\sup_{\partial\Omega} |\nabla u|$ with $\sup_{\Omega} |u|$

III Estimation of $\sup_{\Omega} |\nabla u|$ with $\sup_{\Omega} |u|$ and $\sup_{\partial\Omega} |\nabla u|$

The first two steps are done by constructing suitable barriers and the proof of the third step is based on a method due to Korevaar and Simon.

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Asymptotic Dirichlet problem for f-minimal graphs

Let $\theta\colon \partial_\infty M \to \mathbb{R}$ be continuous and $f\colon M \times \mathbb{R} \to \mathbb{R}$ be a smooth function. The asymptotic Dirichlet problem is to find $u \in C(\bar{M}) \cap C^\infty(M)$ such that

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We are looking for f-minimal submanifold $\Sigma_u \subset M \times \mathbb{R}$ which is the graph of u with "asymptotic boundary"

$$\{(x, \theta(x)) : x \in \partial_\infty M\} \subset \partial_\infty M \times \mathbb{R}.$$

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Theorem 10 (Casteras-H-Holopainen (2016))

Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Assume that there exist constants $\phi > 1$, $\varepsilon > 0$, $k > 0$ and $R_0 > 0$ such that

$$-r(x)^{2(\phi-2)-\varepsilon} \leq K(P_x) \leq -\frac{\phi(\phi-1)}{r(x)^2}$$

OR

$$-r(x)^{-2-\varepsilon}e^{2kr(x)} \leq K(P_x) \leq -k^2$$

for all 2-dimensional subspaces $P_x \subset T_xM$, with $r(x) \geq R_0$. Furthermore, assume that $f \in C^2(M \times \mathbb{R})$ and $|\bar{\nabla} f|$ decays fast enough. Then the asymptotic Dirichlet problem for the f-minimal graph equation is solvable with any boundary data $\theta \in C(\partial_\infty M)$.

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Idea of the proof

In order to solve the Dirichlet problem in a sequence of geodesic balls $B(o,k)$, $k \in \mathbb{N}$, we note that the mean curvature of the boundary $\partial B(o, r(x))$ satisfies

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The correct boundary values are proved by localising the argument to a suitable truncated cone neighbourhood of a point $x_0 \in \partial_\infty M$. We show that a function

$$\psi = A(R_3^\delta r^{-\delta} + h)$$

is a supersolution in this truncated cone. ψ was first introduced by Holopainen and Vähäkangas in a study of p-harmonic functions.

This is the third point where the decay of $|\bar{\nabla} f|$ is required and corresponding to the curvature upper bounds we require

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar{\nabla} f| = o(r^{-1-\varepsilon}) \quad \text{if} \quad K_M \leq -k^2$$

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Necessity of the curvature lower bound

Remark

To solve the asymptotic Dirichlet problem on general manifolds, it is not enough to assume only curvature upper bound.

Theorem 11 (Holopainen-Ripoll (2015))

There exists a 3-dimensional Cartan-Hadamard manifold M, with $K \leq -1$, such that the asymptotic Dirichlet problem is not solvable for any continuous (nonconstant) boundary data.

Case of harmonic functions by Ancona (probabilistic methods) in 1994 and by Borbély (analytic methods) in 1998, p-harmonic functions by Holopainen (generalised Borbély's example) in 2016.

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Optimality of the upper bound

Definition 12

M has asymptotically non-negative sectional curvature if there exists a continuous decreasing function $\lambda\colon [0,\infty) \to [0,\infty)$ such that

$$\int_0^\infty s\lambda(s)\,ds < \infty,$$

and that $K_M(P_x) \geq -\lambda\big(d(o,x)\big)$ at any point $x \in M$

Example 13

If (outside a compact set)

$$K_M(P_x) \geq -\frac{C}{d(o,x)^2\big(\log d(o,x)\big)^{1+\varepsilon}}, \quad C,\varepsilon > 0,$$

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Theorem 14 (Casteras-H-Holopainen (2017))

Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. If $u \colon M \to \mathbb{R}$ is a solution to the minimal graph equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0$$

that is bounded from below and has at most linear growth, then it must be a constant. In particular, if M is a Cartan-Hadamard manifold with asymptotically non-negative sectional curvature, the asymptotic Dirichlet problem is not solvable.

It is worth pointing out that we do not assume the Ricci curvature to be non-negative.

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It is worth pointing out that we do not assume the Ricci curvature to be non-negative.

Optimality of the upper bound

Corollary 15

Let M be a complete Riemannian manifold with only one end and assume that the sectional curvatures of M satisfy

$$K(P_x) \geq -\frac{C}{r(x)^2\big(\log r(x)\big)^{1+\varepsilon}}$$

for sufficiently large $r(x)$ and for any $C > 0$ and $\varepsilon > 0$. Then any solution $u\colon M \to [a,\infty)$ with at most linear growth to the minimal graph equation must be constant.

The proof of Theorem 14 is based on a uniform gradient estimate:

Proposition

Assume that the sectional curvature of M has a lower bound $K(P_x) \geq -K_0^2$ for all $x \in B(p,R)$ for some constant $K_0 = K_0(p,R) \geq 0$. Let u be a positive solution to the minimal graph equation in $B(p,R) \subset M$. Then

$$|\nabla u(p)| \leq \left(\frac{2}{\sqrt{3}} + \frac{32u(p)}{R}\right)\left(\exp\left[64u(p)^2\left(\frac{2\psi(R)}{R^2} + \sqrt{\frac{4\psi(R)^2}{R^4} + \frac{(n-1)K_0^2}{64u(p)^2}}\right)\right] + 1\right),$$

where $\psi(R) = (n-1)K_0 R \coth(K_0 R) + 1$ *if* $K_0 > 0$ *and* $\psi(R) = n$ *if* $K_0 = 0$.

"Proof" of the gradient estimate

Define a function $h = \eta W$, where $\eta(x) = g(\varphi(x))$ with $g(t) = e^{Kt} - 1$,

$$\varphi(x) = \left(1 - \frac{u(x)}{4u(p)} - \frac{d(x,p)^2}{R^2}\right)^+,$$

and a constant K that will be specified later.

We may assume that h attains its maximum at a point q, which is outside the cut-locus $C(p)$ of p. The case $q \in C(p)$ was treated in Rosenberg, Schulze, Spruck 2013.

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”Proof” of the gradient estimate

All the computations will be done at point q. We have

$$\nabla^S h = \eta \nabla^S W + W \nabla^S \eta = 0 \tag{4}$$

and since the Hessian of h is non-positive, we obtain, using (4) and a Bochner-type formula,

$$\begin{aligned} 0 \geq \Delta^S h &= W \Delta^S \eta + 2\langle \nabla^S \eta, \nabla^S W \rangle + \eta \Delta^S W \\ &= W \Delta^S \eta + \left(\Delta^S W - \frac{2}{W} |\nabla^S W|^2 \right) \eta \\ &= W \big(\Delta^S \eta + (|A|^2 + \overline{\mathrm{Ric}}(N,N)) \eta \big). \end{aligned}$$

Combining with ith the Ricci lower bound we have

$$\Delta^S \varphi + K |\nabla^S \varphi|^2 \leq \frac{(n-1)K_0^2}{K}. \tag{5}$$

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"Proof" of the gradient estimate

Idea is to estimate the left hand side of (5). Since u is a solution, we have $\Delta^S u = 0$ and

$$\sum_{i=1}^{n} \langle \bar{\nabla}_{e_i} N, e_i \rangle = 0.$$

Using Hessian comparison we obtain

$$\Delta^S \varphi \geq -\frac{2d}{R^2} \sum_{i=1}^{n} \langle \bar{\nabla}_{e_i} \bar{\nabla} d, e_i \rangle - \frac{2}{R^2} \geq -\frac{2\psi(R)}{R^2},$$

with $\psi(R) = (n-1)K_0 R \coth(K_0 R) + 1$ if $K_0 > 0$ and $\psi(R) = n$ if $K_0 = 0$.

"Proof" of the gradient estimate

Idea is to estimate the left hand side of (5). Since u is a solution, we have $\Delta^S u = 0$ and

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Last step is to estimate $|\nabla^S \varphi|^2$ from below. Assuming

$$W(q) > \max\left\{\frac{2}{\sqrt{3}}, \frac{32u(p)}{R}\right\},$$

choosing

$$K = 128u(p)^2\left(\frac{2\psi(R)}{R^2} + \sqrt{\frac{4\psi(R)^2}{R^4} + \frac{(n-1)K_0^2}{64u(p)^2}}\right)$$

and combining with the previous estimate one gets contradiction with (5).
Therefore

$$W(q) \le \max\left\{\frac{2}{\sqrt{3}}, \frac{32u(p)}{R}\right\}$$

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As a corollary we get the following:

If M is a complete manifold with ANSC and u is a solution that is bounded from below and has at most linear growth, then

$$|\nabla u(x)| \leq C, \quad \text{for all } x \in M \setminus B(o, R_0).$$

Proof.

We may assume that $u > 0$. By the assumptions

$$u(x) \leq c\, d(x,o) \quad \text{and} \quad K(P_x) \geq -\frac{c}{d(x,o)^2} \quad \text{for} \quad x \in M \setminus B(o, R_0/2).$$

We apply the gradient estimate for $p \in M \setminus B(o, R_0)$ with $R = d(p,o)/2 \geq R_0/2$:

For the sectional curvature lower bound in $B(p,R)$: $K_0(p,R)^2 \leq c^2/R^2$, $u(p)/R \leq c$, $\psi(R) \leq c$ and $u(p)^2 K_0^2 \leq c$.

Plugging these to the grad. estimate gives the claim.

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□

About the proof of Theorem 14

Denoting

$$A(x) = \frac{1}{\sqrt{1 + |\nabla u|^2}}$$

we see that

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = \operatorname{div} \big(A(x) \nabla u \big) = 0$$

is equivalent to

$$\frac{1}{A(x)} \operatorname{div} \big(A(x) \nabla u \big) = 0.$$

Now we can interpret the minimal graph equation as a weighted Laplace equation Δ_σ with the weight $\sigma = \sqrt{A}$.

By the gradient estimate there exists a constant $c > 0$ s.t. $c \leq \sigma \leq 1$ in $M \setminus B(o, R_0)$ and hence Δ_σ is uniformly elliptic there.

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Global Harnack's inequality (for positive solutions) can be iterated to yield Hölder continuity estimates for all solutions and, furthermore, a Liouville (or Bernstein) type result for solutions with controlled growth:

Corollary 16

Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. Then there exists a constant $\kappa \in (0, 1]$, depending only on n and on the function λ in the (ANSC) condition such that every solution $u : M \to \mathbb{R}$ to the minimal graph equation with

$$\lim_{d(x,o)\to\infty} \frac{|u(x)|}{d(x,o)^{\kappa}} = 0$$

must be constant.

Thank You!